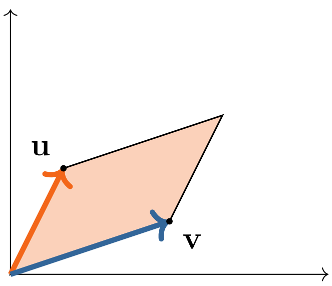
Chapter 6

Volume Forms and Determinants

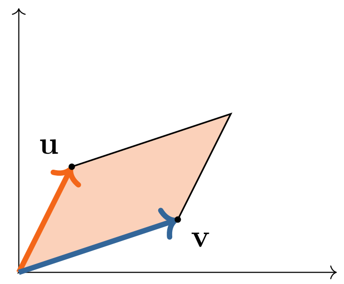
6.1 Motivation

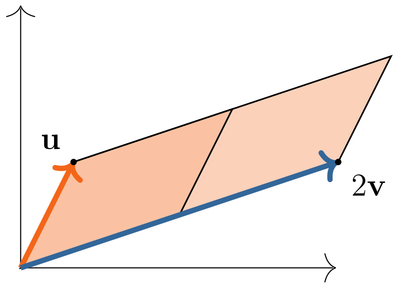
Consider the vector space $V = \mathbb{R}^2$ and let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, as shown in the figure below:



We want to define a function ω, which given two vectors $\mathbf{u}, \mathbf{v} \in V$, returns the area of the parallelogram formed by those two vectors. Thus, we are looking for a function $\omega : V^2 \to \mathbb{R}$, where V^n denotes the space of n-tuples of vectors.

What are the properties we would like ω to satisfy? First, if one of the vectors is multiplied by a scalar, the area should by magnified by the same scalar, as depicted below, where $\mathbf{v}$ has been magnified by a factor of 2:





That is, for every $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ and $a \in \mathbb{R}$,

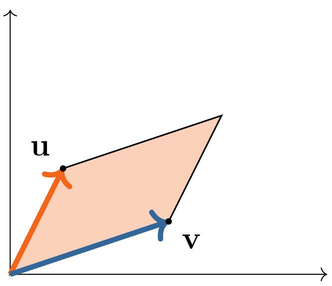
$$\omega(a\,\mathbf{u}, \mathbf{v}) = a\,\omega(\mathbf{u}, \mathbf{v}),$$

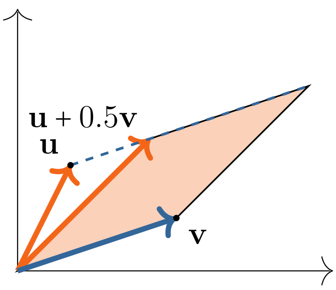
and likewise,

$$\omega(\mathbf{u}, a\,\mathbf{v}) = a\,\omega(\mathbf{u}, \mathbf{v}).$$

Note that if we require this property to hold for every $a \in \mathbb{R}$ we may obtain negative areas; the notion we are looking for is that of a **signed area** (**שטח מסומן**), which is negative or positive depending on the **orientation** (**מגמה**) of the parallelogram. In the case of $\mathbb{R}^2$, $\omega(\mathbf{u}, \mathbf{v})$ whenever the shortest rotation from $\mathbf{u}$ to $\mathbf{v}$ occurs inside the parallelogram (in the above figures the signed area is negative).

The second property we expect ω to satisfy is that if we translate $\mathbf{u}$ along $\mathbf{v}$ (and vice-versa $\mathbf{v}$ along $\mathbf{u}$), then the area doesn't change, as depicted below:

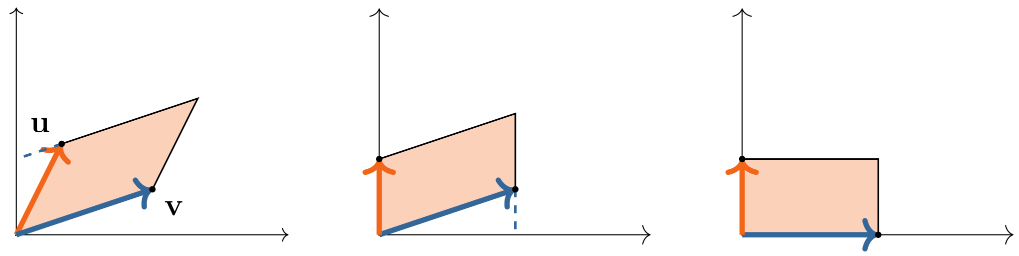




In other words, for every $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ and $a \in \mathbb{R}$,

$$\omega(\mathbf{u} + a\,\mathbf{v}, \mathbf{v}) = \omega(\mathbf{u}, \mathbf{v}) \qquad \text{and} \qquad \omega(\mathbf{u}, \mathbf{v} + a\,\mathbf{u}) = \omega(\mathbf{u}, \mathbf{v}).$$

As we will see, these two properties determine almost uniquely the area function; there always remains a choice of "units", which assigns an area to a reference shape. To understand why, just observe this sequence of transformations, which do not change the area of the parallelogram:



6.2 Volume forms

Definition 6.1 *Let V be an n-dimensional vector space over a field $\mathbb{F}$. A function*

$$\omega : V^n \to \mathbb{F}$$

*is called a **volume form** (תבנית נפח) on V if*

(a) *For every $(\mathbf{v}_1, \dots, \mathbf{v}_n) \in V^n$ and for every $i \neq j$,*

$$\omega(\mathbf{v}_1, \dots, \mathbf{v}_i + \mathbf{v}_j, \dots, \mathbf{v}_n) = \omega(\mathbf{v}_1, \dots, \mathbf{v}_n). \tag{6.1}$$

(b) *For every $a^1, \dots, a^n \in \mathbb{F}$,*

$$\omega(a^1 \mathbf{v}_1, \dots, a^n \mathbf{v}_n) = a^1 \cdots a^n \, \omega(\mathbf{v}_1, \dots, \mathbf{v}_n). \tag{6.2}$$

Note that the function ω returning $0_{\mathbb{F}}$ for every $(\mathbf{v}_1, \dots, \mathbf{v}_n) \in V^n$ satisfies those conditions, i.e., it is a volume form. Such a volume form is called **degenerate** (מנוון).

The following theorem is the central one in this section:

Theorem 6.2 *Let V be an n-dimensional vector space over a field $\mathbb{F}$. For every ordered basis $\mathfrak{B} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$, there exists a unique volume form ω satisfying*

$$\omega(\mathbf{v}_1, \dots, \mathbf{v}_n) = 1_{\mathbb{F}}.$$

We will not prove this theorem right away; for the time being, we will assume that such volume forms exist and examine their properties.

Lemma 6.3 *Let $\omega : V^n \to \mathbb{F}$ be a volume form on an n-dimensional vector space V. Then, for all $(\mathbf{v}_1, \dots, \mathbf{v}_n) \in V^n$ and all $1 \le i \le n$,*

$$\omega(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, 0_V, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n) = 0_{\mathbb{F}}.$$

Proof: This follows from Property (6.2) that

$$\begin{aligned} \omega(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, 0_V, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n) &= \omega(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, 0_{\mathbb{F}}\mathbf{v}_i, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n) \\ &= 0_{\mathbb{F}}\, \omega(\mathbf{v}_1, \dots, \mathbf{v}_n) \\ &= 0_{\mathbb{F}}. \end{aligned}$$

■

Lemma 6.4 *Let $\omega : V^n \to \mathbb{F}$ be a volume form on an n-dimensional vector space V. Then, for all $i \ne j$ and $a \in \mathbb{F}$,*

$$\omega(\mathbf{v}_1, \dots, \mathbf{v}_i + a\,\mathbf{v}_j, \dots, \mathbf{v}_n) = \omega(\mathbf{v}_1, \dots, \mathbf{v}_n).$$

Proof: If $a = 0_{\mathbb{F}}$ then there is nothing to prove. Otherwise,

$$\begin{aligned} a\,\omega(\mathbf{v}_1, \dots, \mathbf{v}_n) &= \omega(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, a\,\mathbf{v}_j, \dots, \mathbf{v}_n) \\ &= \omega(\mathbf{v}_1, \dots, \mathbf{v}_i + a\,\mathbf{v}_j, \dots, a\,\mathbf{v}_j, \dots, \mathbf{v}_n) \\ &= a\,\omega(\mathbf{v}_1, \dots, \mathbf{v}_i + a\,\mathbf{v}_j, \dots, \mathbf{v}_j, \dots, \mathbf{v}_n), \end{aligned}$$

where the first and third equalities follow from (6.2) and the second equality follows from (6.1). Dividing both sides by a, we obtain the required result. ■

Corollary 6.5 *Let $\omega : V^n \to \mathbb{F}$ be a volume form on an n-dimensional vector space V. Then, for all $a^1, \ldots, a^n \in \mathbb{F}$ and $1 \le i \le n$,*

$$\omega\left(\mathbf{v}_1, \ldots, \mathbf{v}_i + \sum_{j \neq i} a^j \mathbf{v}_j, \ldots, \mathbf{v}_n\right) = \omega(\mathbf{v}_1, \ldots, \mathbf{v}_n).$$

Proof: This follows from $(n-1)$ applications of the previous lemma. ■

Corollary 6.6 *Let $\omega : V^n \to \mathbb{F}$ be a volume form on an n-dimensional vector space V. If the sequence of vectors $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is linearly-dependent, then*

$$\omega(\mathbf{v}_1, \ldots, \mathbf{v}_n) = 0_{\mathbb{F}}.$$

Proof: If the vectors are linearly-dependent, then one of the vectors, say $\mathbf{v}_i$, can be written as a linear combination of all the others,

$$\mathbf{v}_i = \sum_{j \neq i} a^j \mathbf{v}_j.$$

Then,

$$\begin{aligned} \omega(\mathbf{v}_1, \ldots, \mathbf{v}_n) &= \omega\left(\mathbf{v}_1, \ldots, \underbrace{0_V + \sum_{j \neq i} a^j \mathbf{v}_j}_{i\text{-th term}}, \ldots, \mathbf{v}_n\right) \\ &= \omega\left(\mathbf{v}_1, \ldots, 0_V, \ldots, \mathbf{v}_n\right) = 0_{\mathbb{F}}. \end{aligned}$$

■

6.3 Volume forms and elementary matrices

Both operations of multiplying one of the vectors by a scalar and adding to one vector a multiple of another vector can be realized by multiplication by an elementary matrix. If we define the elementary matrices

$$(D_k^k(a))_i^j = \begin{cases} 0_{\mathbb{F}} & i \neq j \\ a & i = j = k \\ 1 & i = j \neq k, \end{cases} \tag{6.3}$$

Then,

$$(\mathbf{v}_1, \dots, \mathbf{v}_n)\, D_k^k(a) = (\mathbf{v}_1, \dots, a\, \mathbf{v}_k, \dots, \mathbf{v}_n).$$

Similarly, if we define the elementary matrices

$$(T_k^\ell(a))_i^j = \begin{cases} 1 & i = j \\ a & i = k, j = \ell \\ 0_{\mathbb{F}} & \text{otherwise}, \end{cases} \tag{6.4}$$

Then,

$$(\mathbf{v}_1, \dots, \mathbf{v}_n)\, T_k^\ell(a) = (\mathbf{v}_1, \dots, \mathbf{v}_k + a\, \mathbf{v}_\ell, \dots, \mathbf{v}_n).$$

If follows that for every volume form ω,

$$\omega((\mathbf{v}_1, \dots, \mathbf{v}_n) D_k^k(a)) = a\, \omega(\mathbf{v}_1, \dots, \mathbf{v}_n), \tag{6.5}$$

and

$$\omega((\mathbf{v}_1, \dots, \mathbf{v}_n) T_k^\ell(a)) = \omega(\mathbf{v}_1, \dots, \mathbf{v}_n). \tag{6.6}$$

Example: Let $V = \mathbb{R}^2$ and let

$$\mathfrak{B} = ((1,2),(2,1)) \qquad \text{and} \qquad \mathfrak{C} = ((1,1),(1,-1))$$

be two ordered bases. We have seen that

$$((1,1),(1,-1)) = ((1,2),(2,1)) \begin{bmatrix} 1/3 & -1 \\ 1/3 & 1 \end{bmatrix}.$$

You may verify that

$$\begin{bmatrix} 1/3 & -1 \\ 1/3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix},$$

hence

$$\omega((1,1),(1,-1)) = \omega\left(((1,2),(2,1)) \begin{bmatrix} 1/3 & -1 \\ 1/3 & 1 \end{bmatrix} \right) = \frac{1}{3} \cdot \frac{1}{3} \cdot 6 \cdot \omega((1,2),(2,1)).$$

Thus, the ratio between the volumes associated with these two bases is completely determined by the structure of the transition matrix between those bases. The ratio 3/2 is a property of the transition matrix, which we will identify below as its determinant. ▲▲▲

6.4 Multilinearity and alternation

In this section we are going to examine volume forms from another perspective.

Definition 6.7 *Let V be an n-dimensional vector space over $\mathbb{F}$. A function $f : V^n \to \mathbb{F}$ is called **multilinear** (**מולטילינארית**) if for every $(\mathbf{v}_1, \dots, \mathbf{v}_n) \in V^n$, $\mathbf{w} \in V$ and $a \in \mathbb{F}$,*

$$f(\mathbf{v}_1, \dots, \mathbf{v}_i + \mathbf{w}, \dots, \mathbf{v}_n) = f(\mathbf{v}_1, \dots, \mathbf{v}_n) + f(\mathbf{v}_1, \dots, \mathbf{w}, \dots, \mathbf{v}_n),$$

and

$$f(\mathbf{v}_1, \dots, a\,\mathbf{v}_i, \dots, \mathbf{v}_n) = a\,f(\mathbf{v}_1, \dots, \mathbf{v}_n).$$

Example: Let $\ell^1, \dots, \ell^n \in V^\vee$ be a sequence of linear forms, then the function $f : V^n \to \mathbb{F}$ defined by

$$f(\mathbf{v}_1, \dots, \mathbf{v}_n) = \ell^1(\mathbf{v}_1) \cdots \ell^n(\mathbf{v}_n)$$

is multilinear. ▲▲▲

Example: Let $V = \mathbb{F}^n_{\text{col}}$, then the function $f : V^n \to \mathbb{F}$ defined by

$$f\left(\begin{bmatrix} a_1^1 \\ \vdots \\ a_1^n \end{bmatrix}, \dots, \begin{bmatrix} a_n^1 \\ \vdots \\ a_n^n \end{bmatrix}\right) = a_1^1 a_2^2 \dots a_n^n$$

is multilinear. ▲▲▲

Definition 6.8 *Let V be an n-dimensional vector space over $\mathbb{F}$. A function $f : V^n \to \mathbb{F}$ is called **alternating** (**חילופית**) if for every $(\mathbf{v}_1, \dots, \mathbf{v}_n) \in V^n$ for which $\mathbf{v}_i = \mathbf{v}_j$ with $i \neq j$,*

$$f(\mathbf{v}_1, \dots, \mathbf{v}_n) = 0.$$

Theorem 6.9 *Let V be an n-dimensional vector space over $\mathbb{F}$. Let $f : V^n \to \mathbb{F}$. Then, f is a volume form if and only if it is multilinear and alternating.*

Proof: Suppose first that ω is a volume form. We need to show that it is multilinear and alternating. We will show that it is linear in its first entry, as we can repeat the same argument for all other entries. We need to show that for every $\mathbf{u}, \mathbf{v}, \mathbf{v}_2, \dots, \mathbf{v}_n \in V$,

$$\omega(\mathbf{u} + \mathbf{v}, \mathbf{v}_2, \dots, \mathbf{v}_n) = \omega(\mathbf{u}, \mathbf{v}_2, \dots, \mathbf{v}_n) + \omega(\mathbf{v}, \mathbf{v}_2, \dots, \mathbf{v}_n).$$

By Corollary 6.6, if the vectors $\mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly-dependent, then both sides of this equation are zero. Otherwise, let $\mathbf{v}_1$ be the completion of $\mathbf{v}_2, \dots, \mathbf{v}_n$ into a basis for V, and write

$$\mathbf{u} = a^1 \mathbf{v}_1 + \cdots + a^n \mathbf{v}_n \qquad \text{and} \qquad \mathbf{v} = b^1 \mathbf{v}_1 + \cdots + b^n \mathbf{v}_n.$$

By Properties (6.1),(6.2) of volume forms,

$$\begin{aligned}
\omega(\mathbf{u}, \mathbf{v}_2, \dots, \mathbf{v}_n) &= \omega(a^1 \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) = a^1 \omega(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) \\
\omega(\mathbf{v}, \mathbf{v}_2, \dots, \mathbf{v}_n) &= \omega(b^1 \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) = b^1 \omega(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) \\
\omega(\mathbf{u} + \mathbf{v}, \mathbf{v}_2, \dots, \mathbf{v}_n) &= \omega((a^1 + b^1) \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) = (a^1 + b^1) \omega(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n),
\end{aligned}$$

which proves the additive property. The multiplicative property of multilinearity,

$$\omega(a \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) = a \, \omega(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$$

is a particular case of (6.2).

The alternating property follows from the fact that if $\mathbf{v}_i = \mathbf{v}_j$ for $i \neq j$, then

$$\begin{aligned}
\omega(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, \mathbf{v}_j, \dots, \mathbf{v}_n) &= \omega(\mathbf{v}_1, \dots, \mathbf{v}_i - \mathbf{v}_j, \dots, \mathbf{v}_j, \dots, \mathbf{v}_n) \\
&= \omega(\mathbf{v}_1, \dots, 0_V, \dots, \mathbf{v}_j, \dots, \mathbf{v}_n) = 0_{\mathbb{F}},
\end{aligned}$$

where in the last step we used Lemma 6.3.

Conversely, suppose that ω is multilinear and alternating. Property (6.2) is automatically satisfied. It only remains to prove that

$$\omega(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, \mathbf{v}_j, \dots, \mathbf{v}_n) = \omega(\mathbf{v}_1, \dots, \mathbf{v}_i + \mathbf{v}_j, \dots, \mathbf{v}_j, \dots, \mathbf{v}_n),$$

but this is immediate as, by multilinearity and alternation,

$$\begin{aligned}
\omega(\mathbf{v}_1, \dots, \mathbf{v}_i + \mathbf{v}_j, \dots, \mathbf{v}_j, \dots, \mathbf{v}_n) &= \omega(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, \mathbf{v}_j, \dots, \mathbf{v}_n) \\
&\quad + \omega(\mathbf{v}_1, \dots, \mathbf{v}_j, \dots, \mathbf{v}_j, \dots, \mathbf{v}_n) \\
&= \omega(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, \mathbf{v}_j, \dots, \mathbf{v}_n) + 0_{\mathbb{F}}.
\end{aligned}$$

■

In practice, volume forms are more natural to think of geometrically, and alternating multilinear functions are more convenient to think of algebraically. We have just shown that they are the same.

Proposition 6.10 *Let V be an n-dimensional vector space over $\mathbb{F}$ and let ω be a volume form on V. Then V is anti-symmetric, namely, for every $i \neq j$,*

$$\omega(\mathbf{v}_1,\dots,\mathbf{v}_i,\dots,\mathbf{v}_j,\dots,\mathbf{v}_n) = -\omega(\mathbf{v}_1,\dots,\mathbf{v}_j,\dots,\mathbf{v}_i,\dots,\mathbf{v}_n).$$

Proof: Consider

$$\omega(\mathbf{v}_1,\dots,\mathbf{v}_i+\mathbf{v}_j,\dots,\mathbf{v}_i+\mathbf{v}_j,\dots,\mathbf{v}_n) = 0_{\mathbb{F}},$$

which vanishes by the alternating property of the volume form. Using the multilinearity, two of the terms vanish by the alternating property, remaining with

$$\omega(\mathbf{v}_1,\dots,\mathbf{v}_i,\dots,\mathbf{v}_j,\dots,\mathbf{v}_n) + \omega(\mathbf{v}_1,\dots,\mathbf{v}_j,\dots,\mathbf{v}_i,\dots,\mathbf{v}_n) = 0_{\mathbb{F}}.$$

■

Example: Let $\dim_{\mathbb{F}} V = 2$ and let

$$\mathfrak{B} = (\mathbf{v}_1,\mathbf{v}_2) \qquad \text{and} \qquad \mathfrak{B}^\vee = (\ell^1,\ell^2)$$

be an ordered basis and its dual. Then, the function $\omega : V^2 \to \mathbb{F}$ defined by

$$\omega(\mathbf{u},\mathbf{v}) = \ell^1(\mathbf{u})\ell^2(\mathbf{v}) - \ell^1(\mathbf{v})\ell^2(\mathbf{u})$$

is multilinear and alternating (check it!). In addition,

$$\omega(\mathbf{v}_1,\mathbf{v}_2) = \ell^1(\mathbf{v}_1)\ell^2(\mathbf{v}_2) - \ell^1(\mathbf{v}_2)\ell^2(\mathbf{v}_1) = 1_{\mathbb{F}}.$$

Note that we proved in fact Theorem 6.2 (the existence part) for $n = 2$.

Suppose now that

$$(\mathbf{u},\mathbf{v}) = (\mathbf{v}_1,\mathbf{v}_2)\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

namely,

$$\mathbf{u} = a\,\mathbf{v}_1 + c\,\mathbf{v}_2 \qquad \text{and} \qquad \mathbf{v} = b\,\mathbf{v}_1 + d\,\mathbf{v}_2.$$

Then,

$$\omega(\mathbf{u},\mathbf{v}) = ad - bc,$$

which should ring a bell. This is the what we called the **determinant** of the matrix. ▲▲▲

In view of Theorem 6.9, we can replace Theorem 6.2 by the equivalent:

Theorem 6.11 *Let V be an n-dimensional vector space over a field $\mathbb{F}$. For every ordered basis $\mathfrak{B} = (\mathbf{v}_1,\dots,\mathbf{v}_n)$, there exists a unique multilinear alternating function $\omega : V^n \to \mathbb{F}$ satisfying*

$$\omega(\mathbf{v}_1,\dots,\mathbf{v}_n) = 1_{\mathbb{F}}.$$

Proof: The proof is by induction on $n = \dim_{\mathbb{F}} V$. Take first $n = 1$ and let $\mathfrak{B} = (\mathbf{v}_1)$ be a basis for V. The linear form $\omega : V \to \mathbb{F}$ satisfying $\omega(\mathbf{v}_1) = 1$ is (mutli)linear, alternating (in an empty sense) and normalized. It is unique as there exists a unique linear form that is normalized (the linear forms form a 1-dimensional vector space. hence are all proportional to ω).

Assume that the statement holds for $\dim_{\mathbb{F}} V = n-1$ and let $\dim_{\mathbb{F}} V = n$. Let

$$\mathfrak{B} = (\mathbf{v}_1,\dots,\mathbf{v}_n)$$

be a basis for V and define

$$L = \operatorname{Span}\{\mathbf{v}_1\} \qquad \text{and} \qquad H = \operatorname{Span}\{\mathbf{v}_2,\dots,\mathbf{v}_n\}$$

be linear subspaces of V. We note that $V = L \oplus H$, with $\dim_{\mathbb{F}} L = 1$ and $\dim_{\mathbb{F}} H = n-1$. By the inductive assumption, there exists a unique multilinear alternating function $\omega_H : H^{n-1} \to \mathbb{F}$ satisfying

$$\omega_H(\mathbf{v}_2,\dots,\mathbf{v}_n) = 1_{\mathbb{F}}.$$

Denote by $p_L : V \to V$ and $p_H : V \to V$ the projections on L and H parallel to H and L, respectively. Every vector $\mathbf{u} \in V$ has a unique decomposition

$$\mathbf{u} = \lambda(\mathbf{u})\,\mathbf{v}_1 + p_H(\mathbf{u}),$$

where $\lambda : L \to \mathbb{F}$ is the function satisfying

$$p_L(\mathbf{u}) = \lambda(\mathbf{u})\,\mathbf{v}_1.$$

Note also that we can think of p_H as a linear transformation $V \to H$. Both functions λ and p_H are linear transformations (λ is a linear form).

We now define a function $\omega : V^n \to \mathbb{F}$ as follows,

$$\omega(\mathbf{u}_1,\dots,\mathbf{u}_n) = \sum_{j=1}^{n}(-1)^{j+1}\lambda(\mathbf{u}_j)\,\omega_H(p_H(\mathbf{u}_1),\dots,\widehat{p_H(\mathbf{u}_j)},\dots,p_H(\mathbf{u}_n)),$$

where the "hat" over the j-th term means that this term has been omitted. We now show that ω is a normalized, alternating multilinear function. Let's write it more explicitly.

$$\begin{aligned}\omega(\mathbf{u}_1,\dots,\mathbf{u}_n) &= \lambda(\mathbf{u}_1)\,\omega_H(p_H(\mathbf{u}_2),\dots,p_H(\mathbf{u}_n))\\ &\quad - \lambda(\mathbf{u}_2)\,\omega_H(p_H(\mathbf{u}_1),p_H(\mathbf{u}_3),\dots,p_H(\mathbf{u}_n))\\ &\quad + \lambda(\mathbf{u}_3)\,\omega_H(p_H(\mathbf{u}_1),p_H(\mathbf{u}_2),p_H(\mathbf{u}_4),\dots,p_H(\mathbf{u}_n))\\ &\quad \mp \dots\\ &\quad + (-1)^{n+1}\lambda(\mathbf{u}_n)\,\omega_H(p_H(\mathbf{u}_1),p_H(\mathbf{u}_2),\dots,p_H(\mathbf{u}_{n-1})).\end{aligned}$$

The function ω is multilinear: each of the terms in the sum is linear in each of the $\mathbf{u}_j$'s, either because λ is linear, or because p_H is linear and ω_H is multilinear. The function ω is also alternating. Suppose, for example, that $\mathbf{u}_1 = \mathbf{u}_2$. In all of the summands but two, $\mathbf{u}_1$ and $\mathbf{u}_2$ are arguments of ω_H, which is alternating, hence these terms vanish. Remain two terms, which in this case are

$$\begin{aligned}&\lambda(\mathbf{u}_1)\,\omega_H(p_H(\mathbf{u}_2),p_H(\mathbf{u}_3),\dots,p_H(\mathbf{u}_n))\\ &\qquad - \lambda(\mathbf{u}_2)\,\omega_H(p_H(\mathbf{u}_1),p_H(\mathbf{u}_3),\dots,p_H(\mathbf{u}_n)) = 0_{\mathbb{F}}.\end{aligned}$$

You may convince yourself that this would happen whenever $\mathbf{u}_i = \mathbf{u}_j$ for $i \neq j$. As for the normalization, since $\lambda(\mathbf{v}_i) = 0_{\mathbb{F}}$ and $p_H(\mathbf{v}_i) = \mathbf{v}_i$ for all $i \geq 2$,

$$\omega(\mathbf{v}_1,\dots,\mathbf{v}_n) = \lambda(\mathbf{v}_1)\,\omega_H(p_H(\mathbf{v}_2),\dots,p_H(\mathbf{v}_n)) = 1_{\mathbb{F}}\cdot 1_{\mathbb{F}} = 1_{\mathbb{F}}.$$

We have thus proved that ω is a volume form on V.

It remains to prove the uniqueness. Let $\eta : V^n \to \mathbb{F}$ be a volume form on V satisfying

$$\eta(\mathbf{v}_1,\dots,\mathbf{v}_n) = 1_{\mathbb{F}}.$$

Let $(\mathbf{u}_1, \ldots, \mathbf{u}_n) \in V^n$. If this sequence of vectors is linearly-dependent, then

$$\eta(\mathbf{u}_1, \ldots, \mathbf{u}_n) = 0_{\mathbb{F}} = \omega(\mathbf{u}_1, \ldots, \mathbf{u}_n).$$

Otherwise, $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ is a basis, and there exists an invertible matrix $P \in \mathrm{GL}_n(\mathbb{F})$ such that

$$(\mathbf{u}_1, \ldots, \mathbf{u}_n) = (\mathbf{v}_1, \ldots, \mathbf{v}_n)P.$$

Such a P can be written as a product of elementary matrices of type $D_k^k(a)$ and $T_k^\ell(a)$. By (6.5) and (6.6),

$$\begin{aligned}\omega((\mathbf{v}_1, \ldots, \mathbf{v}_n)D_k^k(a)) &= a\,\omega(\mathbf{v}_1, \ldots, \mathbf{v}_n)\\ &= a\\ &= a\,\eta(\mathbf{v}_1, \ldots, \mathbf{v}_n)\\ &= \eta((\mathbf{v}_1, \ldots, \mathbf{v}_n)D_k^k(a)),\end{aligned}$$

and

$$\begin{aligned}\omega((\mathbf{v}_1, \ldots, \mathbf{v}_n)T_k^\ell(a)) &= \omega(\mathbf{v}_1, \ldots, \mathbf{v}_n)\\ &= 1_{\mathbb{F}}\\ &= \eta(\mathbf{v}_1, \ldots, \mathbf{v}_n)\\ &= \eta((\mathbf{v}_1, \ldots, \mathbf{v}_n)T_k^\ell(a)).\end{aligned}$$

Proceeding inductively

$$\omega((\mathbf{v}_1, \ldots, \mathbf{v}_n)P) = \eta((\mathbf{v}_1, \ldots, \mathbf{v}_n)P),$$

proving that $\omega = \eta$ for all entries $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$. ■

Exercises

(easy) 6.1 Let V be a vector space over $\mathbb{F}$ and let $k \in \mathbb{N}$ (not necessarily the dimension on V). We denote by $\mathrm{Mult}(k, V, \mathbb{F})$ the set of functions $f : V^k \to \mathbb{F}$ that are multilinear (it is a subspace of $\mathrm{Func}(V^k, \mathbb{F})$). Show that $\mathrm{Mult}(k, V, \mathbb{F})$ is a vector space over $\mathbb{F}$.

(intermediate) 6.2 Let V be a three-dimensional vector space over a field $\mathbb{F}$. Let

$$\mathfrak{B} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \qquad \text{and} \qquad \mathfrak{B}^\vee = (\ell^1, \ell^2, \ell^3)$$

be a basis for V and its dual. Consider the function $f : V^3 \to \mathbb{F}$ by

$$\begin{aligned} f(\mathbf{u},\mathbf{v},\mathbf{w}) &= \ell^1(\mathbf{u})(\ell^2(\mathbf{v})\ell^3(\mathbf{w}) - \ell^3(\mathbf{v})\ell^2(\mathbf{w})) \\ &\quad - \ell^2(\mathbf{u})(\ell^1(\mathbf{v})\ell^3(\mathbf{w}) - \ell^3(\mathbf{v})\ell^1(\mathbf{w})) \\ &\quad + \ell^3(\mathbf{u})(\ell^1(\mathbf{v})\ell^2(\mathbf{w}) - \ell^2(\mathbf{v})\ell^1(\mathbf{w})). \end{aligned}$$

Show that f is a normalized volume form on V.

(intermediate) 6.3 Let V be a four-dimensional vector space over a field $\mathbb{F}$. Let

$$\mathfrak{B} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \qquad \text{and} \qquad \mathfrak{B}^\vee = (\ell^1, \ell^2, \ell^3, \ell^4)$$

be a basis for V and its dual. Write using the dual basis a normalized volume form on V.

6.5 Determinants

Let V be an n-dimensional vector space over a field $\mathbb{F}$. Let $\mathfrak{B} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ be an ordered basis and let ω be a volume form on V. By the definition of a basis, every $(\mathbf{u}_1, \dots, \mathbf{u}_n) \in V^n$ has a unique representation as

$$(\mathbf{u}_1, \dots, \mathbf{u}_n) = (\mathbf{v}_1, \dots, \mathbf{v}_n)A$$

for some matrix $A \in M_n(\mathbb{F})$. That is,

$$\omega(\mathbf{u}_1, \dots, \mathbf{u}_n) = \omega((\mathbf{v}_1, \dots, \mathbf{v}_n)A).$$

The right-hand side only depends on the matrix A. Consider then the function

$$f(A) = \frac{\omega((\mathbf{v}_1, \dots, \mathbf{v}_n)A)}{\omega(\mathbf{v}_1, \dots, \mathbf{v}_n)}.$$

We note that it satisfies the following properties:

(a) If $A = I_n$, then $f(A) = 1_{\mathbb{F}}$.

(b) If

$$(\mathbf{u}_1, \dots, \mathbf{u}_n) = (\mathbf{v}_1, \dots, \mathbf{v}_n)A,$$

and A has two columns that are identical, say,

$$\mathrm{Col}_i(A) = \mathrm{Col}_j(A),$$

then $\mathbf{u}_i = \mathbf{u}_j$, hence $\omega(\mathbf{u}_1, \dots, \mathbf{u}_n) = 0_{\mathbb{F}}$. It follows that $f(A) = 0_{\mathbb{F}}$.

(c) By the distributivity of matrix multiplication,

$$(\mathbf{v}_1,\ldots,\mathbf{v}_n)\begin{bmatrix} a_1^1 & \ldots & b^1+c^1 & \ldots & a_n^1 \\ a_1^2 & \ldots & b^2+c^2 & \ldots & a_n^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^n & \ldots & b^n+c^n & \ldots & a_n^n \end{bmatrix} = (\mathbf{v}_1,\ldots,\mathbf{v}_n)\begin{bmatrix} a_1^1 & \ldots & b^1 & \ldots & a_n^1 \\ a_1^2 & \ldots & b^2 & \ldots & a_n^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^n & \ldots & b^n & \ldots & a_n^n \end{bmatrix} + (\mathbf{v}_1,\ldots,\mathbf{v}_n)\begin{bmatrix} a_1^1 & \ldots & c^1 & \ldots & a_n^1 \\ a_1^2 & \ldots & c^2 & \ldots & a_n^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^n & \ldots & c^n & \ldots & a_n^n \end{bmatrix},$$

where the b's and c's are in the i-th column. This is an equation involving three elements of V^n, which all have the same entries except for the i-th entry, where the i-th entry on left-hand side is the sum of the i-th entries on the right-hand side. By the multilinearity of volume forms,

$$\omega\left((\mathbf{v}_1,\ldots,\mathbf{v}_n)\begin{bmatrix} a_1^1 & \ldots & b^1+c^1 & \ldots & a_n^1 \\ a_1^2 & \ldots & b^2+c^2 & \ldots & a_n^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^n & \ldots & b^n+c^n & \ldots & a_n^n \end{bmatrix}\right) = \omega\left((\mathbf{v}_1,\ldots,\mathbf{v}_n)\begin{bmatrix} a_1^1 & \ldots & b^1 & \ldots & a_n^1 \\ a_1^2 & \ldots & b^2 & \ldots & a_n^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^n & \ldots & b^n & \ldots & a_n^n \end{bmatrix}\right) + \omega\left((\mathbf{v}_1,\ldots,\mathbf{v}_n)\begin{bmatrix} a_1^1 & \ldots & c^1 & \ldots & a_n^1 \\ a_1^2 & \ldots & c^2 & \ldots & a_n^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^n & \ldots & c^n & \ldots & a_n^n \end{bmatrix}\right).$$

Dividing both sides by $\omega(\mathbf{v}_1,\ldots,\mathbf{v}_n)$ we obtain

$$f\left(\begin{bmatrix} a_1^1 & \ldots & b^1+c^1 & \ldots & a_n^1 \\ a_1^2 & \ldots & b^2+c^2 & \ldots & a_n^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^n & \ldots & b^n+c^n & \ldots & a_n^n \end{bmatrix}\right) = f\left(\begin{bmatrix} a_1^1 & \ldots & b^1 & \ldots & a_n^1 \\ a_1^2 & \ldots & b^2 & \ldots & a_n^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^n & \ldots & b^n & \ldots & a_n^n \end{bmatrix}\right) + f\left(\begin{bmatrix} a_1^1 & \ldots & c^1 & \ldots & a_n^1 \\ a_1^2 & \ldots & c^2 & \ldots & a_n^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^n & \ldots & c^n & \ldots & a_n^n \end{bmatrix}\right)$$

(d) The n-tuple

$$(\mathbf{v}_1,\dots,\mathbf{v}_n)\begin{bmatrix} a_1^1 & \dots & c\,a_i^1 & \dots & a_n^1 \\ a_1^2 & \dots & c\,a_i^2 & \dots & a_n^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^n & \dots & c\,a_i^n & \dots & a_n^n \end{bmatrix}$$

differs from

$$(\mathbf{v}_1,\dots,\mathbf{v}_n)\begin{bmatrix} a_1^1 & \dots & a_i^1 & \dots & a_n^1 \\ a_1^2 & \dots & a_i^2 & \dots & a_n^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^n & \dots & a_i^n & \dots & a_n^n \end{bmatrix}$$

is the i-th entry, which is c times larger. By the homogeneity of the volume form,

$$\omega\left((\mathbf{v}_1,\dots,\mathbf{v}_n)\begin{bmatrix} a_1^1 & \dots & c\,a_i^1 & \dots & a_n^1 \\ a_1^2 & \dots & c\,a_i^2 & \dots & a_n^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^n & \dots & c\,a_i^n & \dots & a_n^n \end{bmatrix}\right) = c\,\omega\left((\mathbf{v}_1,\dots,\mathbf{v}_n)\begin{bmatrix} a_1^1 & \dots & a_i^1 & \dots & a_n^1 \\ a_1^2 & \dots & a_i^2 & \dots & a_n^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^n & \dots & a_i^n & \dots & a_n^n \end{bmatrix}\right).$$

Dividing both sides by $\omega(\mathbf{v}_1,\dots,\mathbf{v}_n)$ we obtain

$$f\left(\begin{bmatrix} a_1^1 & \dots & c\,a_i^1 & \dots & a_n^1 \\ a_1^2 & \dots & c\,a_i^2 & \dots & a_n^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^n & \dots & c\,a_i^n & \dots & a_n^n \end{bmatrix}\right) = c\,f\left(\begin{bmatrix} a_1^1 & \dots & a_i^1 & \dots & a_n^1 \\ a_1^2 & \dots & a_i^2 & \dots & a_n^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^n & \dots & a_i^n & \dots & a_n^n \end{bmatrix}\right).$$

Let's summarize: given an n-tuple of vectors $(\mathbf{v}_1,\dots,\mathbf{v}_n)$,

$$\omega((\mathbf{v}_1,\dots,\mathbf{v}_n)A) = f(A)\,\omega(\mathbf{v}_1,\dots,\mathbf{v}_n),$$

where the function function $f : M_n(\mathbb{F}) \to \mathbb{F}$ satisfies the following properties:

(a) It is column-wise multilinear.

(b) It is column-wise alternating, i.e., $f(A) = 0_{\mathbb{F}}$ if A has two identical columns.

(c) $f(I_n) = 1_{\mathbb{F}}$.

Definition 6.12 *A function $f : M_n(\mathbb{F}) \to \mathbb{F}$ is called a* ***determinant*** *(דטרמיננטה) if* *(a) It is* ***column-wise multilinear****.*

(b) It is ***column-wise alternating****, i.e.,* $f(A) = 0_{\mathbb{F}}$ *if* A *has two identical columns.*

(c) It is ***normalized****, i.e.,* $f(I_n) = 1_{\mathbb{F}}$*.*

Proposition 6.13 *For every field* $\mathbb{F}$ *and* $n \in \mathbb{N}$ *there exists a unique determinant function* $f : M_n(\mathbb{F}) \to \mathbb{F}$*. We denote this function either by* $A \mapsto \det A$ *or by* $A \mapsto |A|$*.*

Proof: If we view a matrix $A \in M_n(\mathbb{F})$ as a sequence of column-vectors,

$$A = (\mathrm{Col}_1(A), \dots, \mathrm{Col}_n(A)),$$

then the determinant can be viewed as a function

$$f : V^n \to \mathbb{F} \qquad \text{where} \qquad V = \mathbb{F}^n_{\mathrm{col}}.$$

The requirements on f are precisely that it is a volume form normalized such that

$$f(\mathbf{e}_1, \dots, \mathbf{e}_n) = 1_{\mathbb{F}}.$$

By Theorem 6.2 such a function exists and is unique. ∎

Corollary 6.14 *Let* $\omega : V^n \to \mathbb{F}$ *be a volume form on a vector space* V*. For every* n*-tuple* $(\mathbf{v}_1, \dots, \mathbf{v}_n) \in V^n$ *and matrix* $A \in M_n(\mathbb{F})$*,*

$$\omega((\mathbf{v}_1, \dots, \mathbf{v}_n)A) = \det(A)\, \omega(\mathbf{v}_1, \dots, \mathbf{v}_n).$$

We have thus obtained a means for calculating the volume of every n-tuple of vectors given its value for a basis, assuming we know how to calculate the determinant of a matrix.

Exercises

(easy) 6.4 Let $A \in M_n(\mathbb{F})$ and $\lambda \in \mathbb{F}$. Show that

$$\det(\lambda A) = \lambda^n \det(A).$$

(easy) 6.5 Let $A \in M_n(\mathbb{F})$ be such that its n-th column is a linear combination of the other columns. Show that

$$\det A = 0_{\mathbb{F}}.$$

(easy) 6.6 Let $A \in M_n(\mathbb{F})$ be such that $a_i^j = 0_{\mathbb{F}}$ for all $i < j$. Show that

$$\det A = a_1^1 a_2^2 \dots a_n^n.$$

(intermediate) 6.7 In each of the following items is given a function $f : M_3(\mathbb{R}) \to \mathbb{R}$. Determine whether it is (a) columns-wise multilinear, (b) linear or (c) neither:

(a) $f(A) = 1_{\mathbb{F}}$.
(b) $f(A) = 0_{\mathbb{F}}$.
(c) $f(A) = a_1^1 + a_2^2 + a_3^3$.
(d) $f(A) = a_1^1 a_1^1 + 2a_1^1 a_2^2$.
(e) $f(A) = -a_1^1 a_2^1 a_3^3$.
(f) $f(A) = a_2^1 a_3^2 a_1^3 + a_3^1 a_1^2 a_2^3$.

(easy) 6.8 Show that

$$\det \begin{bmatrix} s_1 a + t_1 & s_2 a + t_2 & s_3 a + t_3 \\ s_1 b + t_1 & s_2 b + t_2 & s_3 b + t_3 \\ s_1 c + t_1 & s_2 c + t_2 & s_3 c + t_3 \end{bmatrix} = 0_{\mathbb{F}}$$

for all $a, b, c, s_1, s_2, s_3, t_1, t_2, t_3 \in \mathbb{F}$.

(easy) 6.9 Let $A, B \in M_3(\mathbb{R})$ be given by

$$A = \begin{bmatrix} a^1 & b^1 & c^1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} a^1 - 4b^1 + 9c^1 & 2b^1 & 3c^1 \\ a^2 - 4b^2 + 9c^2 & 2b^2 & 3c^2 \\ a^3 - 4b^3 + 9c^3 & 2b^3 & 3c^3 \end{bmatrix}.$$

What is $\det(B)$ if $\det(A) = 3/2$?

(intermediate) 6.10 (a) Find a function $f : M_3(\mathbb{R})$ which is multilinear with respect to its columns, alternating but not normalized.

(b) Find a function $f : M_3(\mathbb{R})$ which is alternating, normalized, but not multilinear with respect to its columns.

(c) Find a function $f : M_3(\mathbb{R})$ which is multilinear with respect to its columns, normalized, but not alternating.

6.6 Calculating determinants

The determinant of a matrix is defined via three properties: column-wise multilinearity, column-wise alternation, and normalization. In this section we turn these defining properties into an algorithm for calculating determinants.

Proposition 6.15 *Let $D_k^k(a)$ and $T_k^\ell(a)$ be the elementary matrices defined by* (6.3) *and* (6.4)*. Then, for all $A \in M_n(\mathbb{F})$,*

$$\det(A\,D_k^k(a)) = a\,\det(A) \qquad \textit{and} \qquad \det(A\,T_k^\ell(a)) = \det(A).$$

In particular,

$$\det(D_k^k(a)) = a \qquad \textit{and} \qquad \det(T_k^\ell(a)) = 1_{\mathbb{F}},$$

so that for every elementary matrix E

$$\det(A\,E) = \det(E)\,\det(A). \tag{6.7}$$

Proof: This is an immediate consequence of the multilinearity and alternation of the determinant. But we can also look at it differently. Let ω be a volume form on an n-dimensional vector space V over $\mathbb{F}$. By (6.5) and (6.6), with $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ replaced by $(\mathbf{v}_1, \dots, \mathbf{v}_n)A$,

$$\omega((\mathbf{v}_1, \dots, \mathbf{v}_n)AD_k^k(a)) = a\,\omega((\mathbf{v}_1, \dots, \mathbf{v}_n)A),$$

and

$$\omega((\mathbf{v}_1, \dots, \mathbf{v}_n)AT_k^\ell(a)) = \omega((\mathbf{v}_1, \dots, \mathbf{v}_n)A).$$

Dividing by $\omega(\mathbf{v}_1, \dots, \mathbf{v}_n)$, we obtained the desired result. ■

Corollary 6.16 *Let $E_1, \dots, E_n$ be a sequence of elementary matrices. Then,*

$$\det(E_1 \dots E_n) = \det(E_1) \cdots \det(E_n).$$

Proof: Apply (6.7) $(n-1)$ times. ■

Proposition 6.17 *Let $A \in M_n(\mathbb{F})$. Then, $A \in \mathrm{GL}_n(\mathbb{F})$ if and only if $\det A \neq 0$.*

Proof: If $A \in \mathrm{GL}_n(\mathbb{F})$, then it is a product of elementary matrices,

$$A = E_1 \cdots E_n,$$

and since $\det E_i \neq 0$ for all i, it follows from the previous corollary that

$$\det A = \det(E_1) \cdots \det(E_n) \neq 0_{\mathbb{F}}.$$

Conversely, if A is not invertible, then it has a column linearly-dependent on the other columns, hence $\det A = 0_{\mathbb{F}}$. ■

Proposition 6.18 *Let $A, B \in M_n(\mathbb{F})$. Then,*

$$\det(AB) = \det(A)\det(B).$$

Proof: If either A or B are not invertible, then AB is not invertible and both sides of the equation vanish. Otherwise, both A and B can be written as products of elementary matrices,

$$A = E_1 \cdots E_n \qquad \text{and} \qquad B = F_1 \cdots F_k.$$

Then,

$$\det(AB) = \underbrace{\det(E_1) \cdots \det(E_n)}_{\det A} \underbrace{\det(F_1) \cdots \det(F_k)}_{\det B}.$$

■

Example: You may verify that

$$\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Hence

$$\det = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} = -5.$$

▲▲▲

Such a means for calculating determinants is not very convenient. A more systematic way hinges on the proof of Theorem 6.11, which we remind, was inductive on n. The determinant of a matrix can be viewed as a column-wise multilinear, column-wise alternating function $\omega : (\mathbb{F}^n_{\text{col}})^n \to \mathbb{F}$, normalized such that $\omega(\mathbf{e}_1, \dots, \mathbf{e}_n) = 1_{\mathbb{F}}$.

Define

$$L = \operatorname{Span}\{\mathbf{e}_1\} \qquad \text{and} \qquad H = \operatorname{Span}\{\mathbf{e}_2, \dots, \mathbf{e}_n\}.$$

Then,

$$p_L\left(\begin{bmatrix} a^1 \\ a^2 \\ \vdots \\ a^n \end{bmatrix}\right) = \begin{bmatrix} a^1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \text{and} \qquad p_H\left(\begin{bmatrix} a^1 \\ a^2 \\ \vdots \\ a^n \end{bmatrix}\right) = \begin{bmatrix} 0 \\ a^2 \\ \vdots \\ a^n \end{bmatrix},$$

so that

$$\lambda\left(\begin{bmatrix} a^1 \\ a^2 \\ \vdots \\ a^n \end{bmatrix}\right) = a^1$$

and we view p_H as a function $\mathbb{F}^n_{\text{col}} \to \mathbb{F}^{n-1}_{\text{col}}$. By the construction in the proof of Theorem 6.11, the volume form ω_H is the unique normalized volume form on $\mathbb{F}^{n-1}_{\text{col}}$, which is nothing but the determinant for $(n-1)\times(n-1)$ matrices. Thus, we obtain the following formula for the determinant,

$$\begin{aligned} \det A &= \sum_{j=1}^n (-1)^{j+1}\lambda(\operatorname{Col}_j(A)) \det(p_H(\operatorname{Col}_1(A)), \dots, p_H(\widehat{\operatorname{Col}_j}(A)), \dots, p_H(\operatorname{Col}_n(A))) \\ &= \sum_{j=1}^n (-1)^{j+1} a^1_j \det(p_H(\operatorname{Col}^1_1(A), \dots, \widehat{\operatorname{Col}^1_j(A)}, \dots, \operatorname{Col}^1_n(A)), \end{aligned} \tag{6.8}$$

where $\mathrm{Col}_j^i(A)$ is the j-th column of A from which the i-entry has been deleted.

Example: For $n = 2$,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a|d| - b|c| = ad - bc.$$

For $n = 3$,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

▲▲▲

Exercises

(intermediate) 6.11 Let $D : M_n(\mathbb{F}) \to \mathbb{F}$ satisfy

$$D(AB) = D(A)\,D(B)$$

for all $A, B \in M_n(\mathbb{F})$.

(a) Show that if $D(I_n) = 0_{\mathbb{F}}$ then D is the zero function.

(b) Show that if $D(I_n) \neq 0_{\mathbb{F}}$ then $D(I_n) = 1_{\mathbb{F}}$ and $D(A) \neq 0_{\mathbb{F}}$ if A is invertible.

(easy) 6.12 Let $A \in M_2(\mathbb{F})$ and let $\lambda \in \mathbb{F}$. Show that

$$\det(\lambda I_2 - A) = \lambda^2 - \lambda \operatorname{tr} A + \det A,$$

where $\operatorname{tr} A$ is the sum of its diagonal terms.

(intermediate) 6.13 Let $A \in M_2(\mathbb{F})$ such that $A^2 = 0$.

(a) Show that $\det A = 0_{\mathbb{F}}$.

(b) Show that $\lambda I_2 - A$ is invertible for every $\lambda \neq 0_{\mathbb{F}}$.

(c) Show that for every $\lambda \in \mathbb{F}$, $\det(\lambda I_2 - A) = \lambda^2$.

6.7 Determinants and transposition

Determinants are invariant under certain column operations; what about invariance under row operations.

Definition 6.19 *Let $A \in M_{m\times n}(\mathbb{F})$. We denote by $A^t \in M_{n\times m}$ its* ***transpose*** *(המטריצה המשוחלפת), given by*

$$(A^t)^j_i = A^i_j.$$

In the next semester you will see why such an operations makes sense.

Example: If

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix},$$

then

$$A^t = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

▲▲▲

Lemma 6.20 *Let $A \in M_{m\times n}(\mathbb{F})$ and let $B \in M_{n\times k}(\mathbb{F})$. Then*

$$(AB)^t = B^t A^t.$$

Proof: Just follows the definitions,

$$(AB)^i_j = \sum_{k=1}^n a^i_k b^k_j \qquad \text{hence} \qquad ((AB)^t)^i_j = \sum_{k=1}^n a^j_k b^k_i.$$

On the other hand

$$(B^t A^t)^i_j = \sum_{k=1}^n (B^t)^i_k (A^t)^k_j = \sum_{k=1}^n b^k_i a^j_k.$$

■

Lemma 6.21 *Let $E \in M_n(\mathbb{F})$ is an elementary matrix, then*

$$\det E^t = \det E.$$

Proof: This follows from the fact that

$$(D_k^k(a))^t = D_k^k(a) \qquad \text{and} \qquad (T_k^\ell(a))^t = T_\ell^k(a).$$

■

Lemma 6.22 *$A \in M_n(\mathbb{F})$ is invertible if and only if A^t is invertible.*

Proof: A is invertible if and only if its columns are linearly-independent and if and only if its rows are linearly-independent. The claim follows by noting that the rows of A are the columns of A^t and vice-versa. ■

Corollary 6.23 *Let $A \in M_n(\mathbb{F})$. Then,*

$$\det A^t = \det A.$$

Proof: If $A \in \mathrm{GL}_n(\mathbb{F})$, then it can be written as a product of elementary matrices,

$$A = E_1 \cdots E_k.$$

By Lemma 6.20,

$$A^t = E_k^t \cdots E_1^t.$$

Combining with Proposition 6.18 and Lemma 6.21,

$$\det A^t = \det E_k^t \cdots \det E_1^t = \det E_1 \cdots \det E_k = \det A.$$

If $A \notin \mathrm{GL}_n(\mathbb{F})$, then $A^t \notin \mathrm{GL}_n(\mathbb{F})$, and

$$\det A = 0_{\mathbb{F}} = \det A^t.$$

■

The implication of this last proposition is that you can evaluate determinants using row operations; for example, the determinant does not change if a multiple of one row is added to another row.

Exercises

(intermediate) 6.14 Find the determinant of the following matrix,

$$\begin{bmatrix} a & b & & \\ c & d & & \\ * & * & e & f \\ * & * & g & h \end{bmatrix},$$

where the asterisks can represent any scalar.

(intermediate) 6.15 Let $A \in \mathbb{F}^n_{\text{col}}$ and let $B = \mathbb{F}^n_{\text{row}}$ for $n > 1$. What can be said about

$$\det(AB)\,?$$

(intermediate) 6.16 Find

$$\det \begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & 0 & 0 & 0 \\ m & n & 0 & 0 & 0 \\ p & q & 0 & 0 & 0 \end{bmatrix}$$

(easy) 6.17 Calculate the determinants of the following matrices:

$$\begin{bmatrix} 9 & 5 & 6 & 4 \\ 7 & 0 & 3 & 0 \\ 2 & 0 & 0 & 0 \\ 8 & 6 & 4 & 7 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 13 & 5 & 14 & 6 & 17 \\ 0 & 0 & 2 & 0 & 0 \\ 11 & 8 & 15 & 10 & 19 \\ 7 & 0 & 0 & 0 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

(intermediate) 6.18 Calculate the determinants of the following $n \times n$ matrices, $n > 2$,

$$\begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 2 & \dots & 1 \\ \vdots & & & \ddots & \vdots \\ 1 & 1 & 1 & \dots & n-1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & \dots & n \\ 2 & 3 & 4 & \dots & n+1 \\ 3 & 4 & 5 & \dots & n+2 \\ \vdots & & & \ddots & \vdots \\ n & n+1 & n+2 & \dots & 2n-1 \end{bmatrix} \quad \begin{bmatrix} a & b & b & \dots & b \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ \vdots & & & \ddots & \vdots \\ b & b & b & \dots & a \end{bmatrix}$$

(intermediate) 6.19 Let $a_1, \dots, a_n \in \mathbb{F}$. Calculate

$$\det \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ a_1 & a_2 & a_3 & \dots & a_n \\ (a_1)^2 & (a_2)^2 & (a_3)^2 & \dots & (a_n)^2 \\ \vdots & & & & \cdots \\ (a_1)^{n-1} & (a_2)^{n-1} & (a_3)^{n-1} & \dots & (a_n)^{n-1} \end{bmatrix}$$

.

(intermediate) 6.20 Let $A, B \in M_3(\mathbb{R})$ find $\det(2A^2B^{-1})$ given than $\det A = 5$ and $\det B = 10$.

(intermediate) 6.21 Let $A, B \in M_3(\mathbb{R})$ find $\det(5AB^3A^{-1}B^{-1})$ given than $\det A \neq 0$ and $\det B = 2$.

(intermediate) 6.22 For each of the following matrices, calculate the determinants and determine for what values of the parameters those matrices are invertible:

$$\begin{bmatrix} 1 & a-2 & -a+1 \\ 0 & 2 & a-1 \\ a & a^2 & a^2-1 \end{bmatrix} \quad \begin{bmatrix} b-3 & -2 \\ 1 & b-6 \end{bmatrix} \quad \begin{bmatrix} c-1 & 4 \\ 2 & c-3 \end{bmatrix} \quad \begin{bmatrix} 1 & d & d^2 \\ d & d^2 & 1 \\ d^2 & 1 & d \end{bmatrix}.$$

(intermediate) 6.23 Let $A \in M_n(\mathbb{R})$ satisfy $A^2 = -A - I_n$.

(a) Show that A is invertible.

(b) Show that $A^3 = I_n$.

(c) Find $\det A$.

(intermediate) 6.24 Let $A, B \in M_n(\mathbb{F})$.

(a) Show that if $AB + B$ is invertible then so is $BA + B$.

(b) Show that if $A^2B - A^2$ is invertible then so is $BA - A$.

(c) Show that if $AB^2 - A$ is invertible then so is $BA - A$.

(d) Show that if $A^2 - B^2$ is invertible and $AB = BA$, then $A + B$ is invertible.

(intermediate) 6.25 Calculate the determinant of the matrix

$$\begin{bmatrix} a^2 & b^2 & c^2 & d^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 & (d+1)^2 \\ (a+2)^2 & (b+2)^2 & (c+2)^2 & (d+2)^2 \\ (a+3)^2 & (b+3)^2 & (c+3)^2 & (d+3)^2 \end{bmatrix}.$$

6.8 Cramer's formula

Let's look back on how we calculated the determinant of a matrix. The construction was inductive, based on the proof of the existence of a normalized volume form. Let's write the formula in a more compact form: First, let's denote by

$$A^{\cancel{i}} \qquad A_{\cancel{j}} \qquad \text{and} \qquad A^{\cancel{i}}_{\cancel{j}}$$

the matrix A with the i-th row removed, the j-column removed and both the i-th row and the j-th column removed. Formula (6.8) for the determinant of a matrix can be written as

$$\det A = \sum_{j=1}^{n} (-1)^{j+1} a^1_j \det A^{\cancel{1}}_{\cancel{j}}. \tag{6.9}$$

This is of course a recursive formula. It is worth recalling why it is correct. The determinant is the unique function $M_n(\mathbb{F}) \to \mathbb{F}$, which is column-wise multilinear, column-wise alternating and satisfying $\det I_n = 1$. (Since it is invariant under transposition, it is also the unique function which is row-wise multilinear, row-wise alternating and satisfying $\det I_n = 1$.)

The fact that the inductive definition (6.9) satisfies these requirements is proved inductively on n. For example, assume that the determinant is alternating for $M_{n-1}(\mathbb{F})$, and let $A \in M_n(\mathbb{F})$ have its k-th column equal to its ℓ-th column. Then, $A^{\cancel{1}}_{\cancel{j}}$ has two identical columns unless $j = k$ or $j = \ell$; that is, unless $j = k$ or $j = \ell$ we have by the inductive assumption that $\det A^{\cancel{1}}_{\cancel{j}} = 0_{\mathbb{F}}$. Thus,

$$\det A = (-1)^{k+1} a^1_k \det A^{\cancel{1}}_{\cancel{k}} + (-1)^{\ell+1} a^1_\ell \det A^{\cancel{1}}_{\cancel{\ell}}.$$

Now $a^1_k = a^1_\ell$ and the matrices $A^{\cancel{1}}_{\cancel{k}}$ and $A^{\cancel{1}}_{\cancel{\ell}}$ are almost identical; they may only differ in the ordering of the columns. If, for example, $\ell = k+1$, then $A^{\cancel{1}}_{\cancel{k}} = A^{\cancel{1}}_{\cancel{\ell}}$, hence

$$\det A = (-1)^{k+1} a^1_k \det A^{\cancel{1}}_{\cancel{k}} + (-1)^{k+2} a^1_k \det A^{\cancel{1}}_{\cancel{k}} = 0_{\mathbb{F}}.$$

If, for example $\ell = k+2$, then $A^{\cancel{1}}_{\cancel{k}}$ and $A^{\cancel{1}}_{\cancel{\ell}}$ differ by the interchange of two columns, which implies that their determinants differ by a sign, i.e.,

$$\det A = (-1)^{k+1} a^1_k \det A^{\cancel{1}}_{\cancel{k}} + (-1)^{k+3} a^1_k (-\det A^{\cancel{1}}_{\cancel{k}}) = 0_{\mathbb{F}}.$$

Keep "playing with this" to convince yourself that it does not matter how far apart k and ℓ are; in either case, the determinant for $n \times n$ matrices is alternating.

Formula (6.9) delineates the first row as "special"; there is of course nothing special about it. We could have selected any row i, and write instead

$$\det A = \sum_{j=1}^{n} (-1)^{j+i} a^i_j \det A^{\cancel{i}}_{\cancel{j}}. \tag{6.10}$$

Example: Take the 3×3 matrix

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 7 & 2 & 1 \\ 9 & 3 & 2 \end{bmatrix}.$$

The term $(-1)^{j+i}$ yields the following pattern,

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}.$$

Take for example the second row, $i = 2$. Then,

$$A^{\cancel{2}}_{\cancel{1}} = \begin{bmatrix} 1 & 3 & 4 \\ 7 & 2 & 1 \\ 9 & 3 & 2 \end{bmatrix} \qquad A^{\cancel{2}}_{\cancel{2}} = \begin{bmatrix} 1 & 3 & 4 \\ 7 & 2 & 1 \\ 9 & 3 & 2 \end{bmatrix} \qquad A^{\cancel{2}}_{\cancel{3}} = \begin{bmatrix} 1 & 3 & 4 \\ 7 & 2 & 1 \\ 9 & 3 & 2 \end{bmatrix}$$

So that

$$\det A = -7 \begin{vmatrix} 3 & 4 \\ 3 & 2 \end{vmatrix} + 2 \begin{vmatrix} 1 & 4 \\ 9 & 2 \end{vmatrix} - 1 \begin{vmatrix} 1 & 3 \\ 9 & 3 \end{vmatrix} = (-7)(-6) + 2(-34) + (-1)(-24) = (-2).$$

▲ ▲ ▲

Since the determinant is invariant under transposition, we could have as well chosen a distinguished column, say the j-th column, and then sum up over all rows

$$\det A = \sum_{i=1}^{n} (-1)^{j+i} a^i_j \det A^{\cancel{i}}_{\cancel{j}}. \tag{6.11}$$

Example: Take the same matrix as in the previous example and take say the third column, $j = 3$. Then,

$$A^{\cancel{1}}_{\cancel{3}} = \begin{bmatrix} 1 & 3 & 4 \\ 7 & 2 & 1 \\ 9 & 3 & 2 \end{bmatrix} \qquad A^{\cancel{2}}_{\cancel{3}} = \begin{bmatrix} 1 & 3 & 4 \\ 7 & 2 & 1 \\ 9 & 3 & 2 \end{bmatrix} \qquad A^{\cancel{3}}_{\cancel{3}} = \begin{bmatrix} 1 & 3 & 4 \\ 7 & 2 & 1 \\ 9 & 3 & 2 \end{bmatrix}$$

So that

$$\det A = 4 \begin{vmatrix} 7 & 2 \\ 9 & 3 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 3 \\ 9 & 3 \end{vmatrix} + 2 \begin{vmatrix} 1 & 3 \\ 7 & 2 \end{vmatrix} = 4 \cdot 3 - (-24) + 2(-19) = (-2).$$

▲▲▲

Next, we relate determinants to the first subject of this course, the solution of linear systems; we focus on the case where the number of equations equals the number of unknowns. Let $A \in M_n(\mathbb{F})$ and $\mathbf{b} \in \mathbb{F}^n_{\text{col}}$, and denote by

$$A_{\cancel{j}\to\mathbf{b}}$$

The matrix in which the j-th column has been replaced by the column matrix $\mathbf{b}$.

Theorem 6.24 (Cramer's formula) *Let A and $\mathbf{b}$ be as above and suppose that $\mathbf{x} \in \mathbb{F}^n_{col}$ satisfies the equation*

$$A\mathbf{x} = \mathbf{b}.$$

The for every $j = 1, \dots, n$,

$$\det A_{\cancel{j}\to\mathbf{b}} = x^j \det A.$$

In particular, if A is invertible, then

$$x^j = \frac{\det A_{\cancel{j}\to\mathbf{b}}}{\det A}.$$

Example: Consider the linear system,

$$\begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

Then,

$$x^1 = \frac{\begin{vmatrix} 4 & 3 \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{4}{(-1)} \qquad \text{and} \qquad x^2 = \frac{\begin{vmatrix} 5 & 4 \\ 2 & 0 \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{(-8)}{(-1)},$$

and indeed,

$$\begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

But let's actually go through the steps of the proof below. We have

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 5x^1 + 3x^2 \\ 2x^1 + x^2 \end{bmatrix}.$$

Hence,

$$\begin{vmatrix} 4 & 3 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 5x^1 + 3x^2 & 3 \\ 2x^1 + x^2 & 1 \end{vmatrix} = \begin{vmatrix} 5x^1 & 3 \\ 2x^1 & 1 \end{vmatrix} = x^1 \begin{vmatrix} 5 & 3 \\ 2 & 1 \end{vmatrix},$$

and

$$\begin{vmatrix} 5 & 4 \\ 2 & 0 \end{vmatrix} = \begin{vmatrix} 5 & 5x^1 + 3x^2 \\ 2 & 2x^1 + x^2 \end{vmatrix} = \begin{vmatrix} 5 & 3x^2 \\ 2 & x^2 \end{vmatrix} = x^2 \begin{vmatrix} 5 & 3 \\ 2 & 1 \end{vmatrix}.$$

▲ ▲ ▲

Proof: The matrix $\mathbf{b}$ satisfies the equation

$$\mathbf{b} = \sum_{i=1}^{n} x^i \operatorname{Col}_i(A).$$

Take $\mathbf{b}$, place it instead of the j-th column of A; then,

$$A_{\cancel{j}\to\mathbf{b}} = \begin{bmatrix} \operatorname{Col}_1(A) & \dots & \sum_{i=1}^n x^i \operatorname{Col}_i(A) & \dots & \operatorname{Col}_n(A) \end{bmatrix},$$

where the sum is ay the j-th column. By the multilinearity of the determinant,

$$\det A_{\cancel{j}\to\mathbf{b}} = \sum_{i=1}^{n} x^i \begin{vmatrix} \operatorname{Col}_1(A) & \dots & \operatorname{Col}_i(A) & \dots & \operatorname{Col}_n(A) \end{vmatrix}.$$

By the alternation of the determinant, all the summands vanish, except for the j-th, i.e.,

$$\det A_{\cancel{j}\to\mathbf{b}} = x^j \det A.$$

■

And with this, we obtain Cramer's formula for the inverse matrix:

Theorem 6.25 (Cramer's formula) *Let* $A \in \mathrm{GL}_n(\mathbb{F})$ *and denote* $B = A^{-1}$. *Then,*

$$b^i_j = (-1)^{j+i} \frac{\det A^{\cancel{j}}_{\cancel{i}}}{\det A}.$$

Proof: Let's verify that this coincides with the known formula for 2×2 matrices. We have

$$(A^{-1})^1_1 = (-1)^2 \frac{\det A^{\cancel{1}}_{\cancel{1}}}{\det A} = \frac{d}{ad - bc} \qquad (A^{-1})^1_2 = (-1)^3 \frac{\det A^{\cancel{2}}_{\cancel{1}}}{\det A} = -\frac{b}{ad - bc}$$

$$(A^{-1})^2_1 = (-1)^3 \frac{\det A^{\cancel{1}}_{\cancel{2}}}{\det A} = -\frac{c}{ad - bc} \qquad (A^{-1})^2_2 = (-1)^4 \frac{\det A^{\cancel{2}}_{\cancel{2}}}{\det A} = \frac{a}{ad - bc},$$

i.e.,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

■

Proof: The matrix B satisfies the equation $AB = I_n$, which we may rewrite as

$$A \begin{bmatrix} \mathrm{Col}_1(B) & \dots & \mathrm{Col}_n(B) \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & \dots & \mathbf{e}_n \end{bmatrix}.$$

That is,

$$A \, \mathrm{Col}_j(B) = \mathbf{e}_j.$$

By Cramer's formula,

$$b^i_j = \frac{\det A_{\cancel{i}\to\mathbf{e}_j}}{\det A}.$$

Consider the numerator. The i-th column of the matrix $A_{\cancel{i}\to\mathbf{e}_j}$ consists of zeros, except for 1 at the j-th row. Hence, by (6.10),

$$\det A_{\cancel{i}\to\mathbf{e}_j} = (-1)^{i+j} \det A^{\cancel{j}}_{\cancel{i}}.$$

■

Exercises

(intermediate) 6.26 Solve the following linear systems over $\mathbb{R}$ using Cramer's formula.

(a)

$$\begin{array}{cccl} X & +2Y & +3Z & = 6 \\ 4X & +5Y & +6Z & = 15 \\ 7X & +8Y & +10Z & = 25. \end{array}$$

(b)

$$\begin{array}{cccl} X & +Y & +Z & = 11 \\ 2X & -6Y & -Z & = 0 \\ 3X & +4Y & +2Z & = 0. \end{array}$$

(c)

$$\begin{array}{cccl} 3X & -2Y & & = 7 \\ & 3Y & -2Z & = 6 \\ -2X & & +3Z & = -1. \end{array}$$

(intermediate) 6.27 Invert the following matrices using Cramer's formula,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 3 & 2 \\ 6 & 0 & 3 \\ 4 & 1 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}.$$

6.9 The determinant of a linear transformation

In our introduction of determinants we essentially proved the following theorem (Corollary 6.14):

Theorem 6.26 *Let V be an n-dimensional vector space over $\mathbb{F}$ and let $\omega : V^n \to \mathbb{F}$ be a volume form on V. Then, for every ordered basis $\mathfrak{B}$ and any matrix $A \in M_n(\mathbb{F})$, writing*

$$\begin{pmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_n \end{pmatrix} = \mathfrak{B}A,$$

we have

$$\omega\begin{pmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_n \end{pmatrix} = \omega(\mathfrak{B}) \det A.$$

This lead to the perhaps surprising corollary:

Corollary 6.27 *For every non-degenerate volume form ω, every basis $\mathfrak{B}$ and every matrix A, the ratio*

$$\frac{\omega(\mathfrak{B}A)}{\omega(\mathfrak{B})}$$

depends neither on the volume form, nor on the basis.

And further,

Corollary 6.28 *Let ω and η be two non-degenerate volume forms on V, then there exists a constant $c \in \mathbb{F}$ such that*

$$\omega = c\eta,$$

i.e., for every $\begin{pmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_n \end{pmatrix}$,

$$\omega\begin{pmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_n \end{pmatrix} = c\eta\begin{pmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_n \end{pmatrix}.$$

Proof: Let $\mathfrak{B}$ be any ordered basis on V and let $A \in M_n(\mathbb{F})$ be the unique matrix satisfying

$$\begin{pmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_n \end{pmatrix} = \mathfrak{B}A.$$

Then,

$$\omega\begin{pmatrix}\mathbf{u}_1 & \dots & \mathbf{u}_n\end{pmatrix} = \omega(\mathfrak{B}A) = \frac{\omega(\mathfrak{B}A)}{\omega(\mathfrak{B})}\omega(\mathfrak{B}) = \frac{\eta(\mathfrak{B}A)}{\eta(\mathfrak{B})}\omega(\mathfrak{B}) = c\,\eta\begin{pmatrix}\mathbf{u}_1 & \dots & \mathbf{u}_n\end{pmatrix},$$

where

$$c = \frac{\omega(\mathfrak{B})}{\eta(\mathfrak{B})}.$$

■

Since all the volume forms are multiples of each other, they are essentially the same; they only differ by a choice of units. This observation yields that an operator on a vector space can be characterized by how much it magnifies volumes:

Theorem 6.29 *Let ω be a non-degenerate volume form on a finitely-generated vector space V. Let $f \in \mathrm{Hom}_{\mathbb{F}}(V,V)$ be a linear transformation. Let $\mathfrak{B} = \begin{pmatrix}\mathbf{v}_1 & \dots & \mathbf{v}_n\end{pmatrix}$ by an ordered basis on V; we denote*

$$f(\mathfrak{B}) = \big(f(\mathbf{v}_1),\dots,f(\mathbf{v}_n)\big).$$

Then, the ratio

$$\frac{\omega(f(\mathfrak{B}))}{\omega(\mathfrak{B})}$$

depends neither on ω nor on $\mathfrak{B}$; it is a sole property of the linear transformation f, which we call the determinant of f.

Proof: Let $A = [f]^{\mathfrak{B}}_{\mathfrak{B}}$, i.e.,

$$f(\mathfrak{B}) = \mathfrak{B}A.$$

Then,

$$\frac{\omega(f(\mathfrak{B}))}{\omega(\mathfrak{B})} = \frac{\omega(\mathfrak{B}A)}{\omega(\mathfrak{B})} = \det A,$$

and the right-hand side depends neither on ω nor on $\mathfrak{B}$. ■

Thus, the determinant of f coincides with the determinant of its representing matrix, *but*, this identity does not depend on the basis relative to which we represent f. This should perhaps not come as a surprise, as if $\mathfrak{C}$ is some other basis, then there exists an invertible matrix P, such that

$$[f]_{\mathfrak{C}}^{\mathfrak{C}} = P^{-1}[f]_{\mathfrak{B}}^{\mathfrak{B}}P.$$

By the properties of the determinant,

$$\det[f]_{\mathfrak{C}}^{\mathfrak{C}} = \det P^{-1} \cdot \det[f]_{\mathfrak{B}}^{\mathfrak{B}} \cdot \det P.$$

Since $\det P^{-1} \det P = \det I_n = 1_{\mathbb{F}}$, we obtain that the determinant of the representing matrix is independent of the representation, i.e., it is an intrinsic property of the transformation.

$$\det[f]_{\mathfrak{C}}^{\mathfrak{C}} = \det[f]_{\mathfrak{B}}^{\mathfrak{B}}$$

Exercises

(intermediate) 6.28 Let V be a finitely-generated vector space and let $\mathfrak{B}$ be an ordered basis for V. Let $S, T \in \mathrm{Hom}_{\mathbb{F}}(V, V)$. Show that

$$\det[T \circ S]_{\mathfrak{B}}^{\mathfrak{B}} = \det[T]_{\mathfrak{B}}^{\mathfrak{B}} \det[S]_{\mathfrak{B}}^{\mathfrak{B}}.$$

(intermediate) 6.29 Let V be a finitely-generated vector space and let $\mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ be ordered bases for V. Show that

(a) $\det[\mathrm{Id}_V]_{\mathfrak{C}}^{\mathfrak{B}} \neq 0$.
(b) $(\det[\mathrm{Id}_V]_{\mathfrak{C}}^{\mathfrak{B}})^{-1} = \det[\mathrm{Id}_V]_{\mathfrak{B}}^{\mathfrak{C}}$.
(c) $\det[\mathrm{Id}_V]_{\mathfrak{D}}^{\mathfrak{B}} = \det[\mathrm{Id}_V]_{\mathfrak{D}}^{\mathfrak{C}} \det[\mathrm{Id}_V]_{\mathfrak{C}}^{\mathfrak{B}}$.

(intermediate) 6.30 Let $A \in M_n(\mathbb{F})$ and define a linear transformation $g : M_n(\mathbb{F}) \to M_n(\mathbb{F})$,

$$g(X) = XA - AX.$$

Show that $\det g = 0_{\mathbb{F}}$.

(intermediate) 6.31 Let $A \in M_n(\mathbb{F})$ and define two linear transformations $L, R : M_n(\mathbb{F}) \to M_n(\mathbb{F})$,

$$L(X) = AX \qquad \text{and} \qquad R(X) = XA.$$

Show that

$$\det L = \det R = (\det A)^n.$$

Hint: Separate the cases $A \in \mathrm{GL}_n(\mathbb{F})$ and $A \notin \mathrm{GL}_n(\mathbb{F})$.