Chapter 3

Integration

3.1 Measurable functions

The fundamental object that measure theory works with is a measurable space, i.e., a set endowed with a structure—a σ-algebra of subsets. Like in many other instances in mathematics, the moment we have **objects**, we define **morphisms** between such objects, which are maps between sets preserving the structure (morphisms between sets are (general) functions, morphisms between vector spaces are linear transformations, morphisms between groups are homomorphisms, morphisms between topological spaces are continuous functions, etc.).

Recall that for every function $f : \mathbb{X} \to \mathbb{Y}$, where $\mathbb{X}$ and $\mathbb{Y}$ are arbitrary sets, its inverse can be interpreted as a map

$$f^{-1} : \mathscr{P}(\mathbb{Y}) \to \mathscr{P}(\mathbb{X}),$$

where for $A \subset \mathbb{Y}$,

$$f^{-1}(A) = \{x \in \mathbb{X} \,:\, f(x) \in A\} \subset \mathbb{X}.$$

This map commutes with set-theoretic operations, namely,

$$f^{-1}\left(\bigcup_\alpha A_\alpha\right) = \bigcup_\alpha f^{-1}(A_\alpha) \qquad f^{-1}\left(\bigcap_\alpha A_\alpha\right) = \bigcap_\alpha f^{-1}(A_\alpha),$$

and

$$f^{-1}(A^c) = \left(f^{-1}(A)\right)^c.$$

Note that in the above relations the collections $\{A_\alpha\}$ need not be countable. Likewise, we may apply f^{-1} to collections of subsets, i.e.,

$$f^{-1} : \mathscr{P}(\mathscr{P}(\mathbb{Y})) \to \mathscr{P}(\mathscr{P}(\mathbb{X})),$$

where for $\mathcal{C} \subset \mathscr{P}(\mathbb{Y})$,

$$f^{-1}(\mathcal{C}) = \{A \subset \mathbb{X} \,:\, \{f(x) \,:\, x \in A\} \in \mathcal{C}\}.$$

The following lemma assets that any function into a measurable spaces **pulls back** a measurable structure on its domain (similarly to how any function into a topological space pulls back a topology on its domain):

Lemma 3.1 Let $\mathbb{X}$ be a set, let $(\mathbb{Y}, \Sigma_\mathbb{Y})$ be a measurable space and let $f : \mathbb{X} \to \mathbb{Y}$. Then, the collection of sets

$$f^{-1}(\Sigma_\mathbb{Y}) = \{f^{-1}(A) \,:\, A \in \Sigma_\mathbb{Y}\} \subset \mathscr{P}(\mathbb{X})$$

is a σ-algebra on $\mathbb{X}$.

Proof: This follows from the fact that f^{-1} commutes with set-theoretic operations. ■

Definition 3.2 Let $(\mathbb{X}, \Sigma_\mathbb{X})$ and $(\mathbb{Y}, \Sigma_\mathbb{Y})$ be measurable spaces. A mapping $f : \mathbb{X} \to \mathbb{Y}$ is called ***measurable*** *(מדידה) if*

$$f^{-1}(A) \in \Sigma_\mathbb{X} \qquad \textit{for every } A \in \Sigma_\mathbb{Y}.$$

That is, if

$$f^{-1}(\Sigma_\mathbb{Y}) \subset \Sigma_\mathbb{X}.$$

Comment: Strictly speaking, measurability is a relation between σ-algebras; we should say that f is $(\Sigma_\mathbb{X}, \Sigma_\mathbb{Y})$-measurable, because $\mathbb{X}$ and $\mathbb{Y}$ may be endowed with multiple σ-algebras.

Example: The finer $\Sigma_\mathbb{X}$ is and the coarser $\Sigma_\mathbb{Y}$ is, the more there are measurable functions $\mathbb{X} \to \mathbb{Y}$. In the extreme cases, if $\Sigma_\mathbb{Y} = \{\varnothing, \mathbb{Y}\}$, then every function $\mathbb{X} \to \mathbb{Y}$ is measurable, and likewise if $\Sigma_\mathbb{X} = \mathscr{P}(\mathbb{X})$. ▲▲▲

✎ *Exercise 3.1* Let $(\mathbb{X}, \Sigma_{\mathbb{X}})$ and $(\mathbb{Y}, \Sigma_{\mathbb{Y}})$ be measurable spaces. In the absence of any other information, which are the only functions $f : \mathbb{X} \to \mathbb{Y}$ that are guaranteed to ve measurable?

Proposition 3.3 *Let $(\mathbb{X}, \Sigma_{\mathbb{X}})$ and $(\mathbb{Y}, \Sigma_{\mathbb{Y}})$ be measurable spaces. Suppose that $\Sigma_{\mathbb{Y}}$ is generated by a collection of sets $\mathcal{E}$. Then, $f : \mathbb{X} \to \mathbb{Y}$ is measurable if and only if*

$$f^{-1}(E) \in \Sigma_{\mathbb{X}} \qquad \text{for every } E \in \mathcal{E},$$

i.e.,

$$f^{-1}(\mathcal{E}) \subset \Sigma_{\mathbb{X}}. \tag{3.1}$$

Proof: The "only if" part is trivial, as if f is measurable, then

$$f^{-1}(\mathcal{E}) \subset f^{-1}(\Sigma_{\mathbb{Y}}) \subset \Sigma_{\mathbb{X}}.$$

For the "if" part, suppose that (3.1) is satisfied. Consider the collection of sets

$$\mathcal{F} = \{A \subset \mathbb{Y} \,:\, f^{-1}(A) \in \Sigma_{\mathbb{X}}\}.$$

This set contains $\mathcal{E}$. It is also a σ-algebra since f^{-1} commutes with set-theoretic operations; for example,

$$A \in \mathcal{F} \quad \Rightarrow \quad f^{-1}(A) \in \Sigma_{\mathbb{X}} \quad \Rightarrow \quad f^{-1}(A^c) = (f^{-1}(A))^c \in \Sigma_{\mathbb{X}} \quad \Rightarrow \quad A^c \in \mathcal{F}.$$

It follows that $\Sigma_{\mathbb{Y}} \subset \mathcal{F}$, i.e.,

$$f^{-1}(A) \in \Sigma_{\mathbb{X}} \qquad \text{for every } A \in \Sigma_{\mathbb{Y}},$$

which by definition means that f is measurable. ∎

Corollary 3.4 *Let $(\mathbb{X}, \tau_{\mathbb{X}})$ and $(\mathbb{Y}, \tau_{\mathbb{Y}})$ be topological spaces endowed with the Borel σ-algebras. Then, every continuous function $f : \mathbb{X} \to \mathbb{Y}$ is measurable.*

Proof: A function $f : \mathbb{X} \to \mathbb{Y}$ is continuous if the pre-image of every open set is open,

$$f^{-1}(E) \in \tau_{\mathbb{X}} \subset \mathscr{B}(\mathbb{X}) \qquad \text{for every } E \in \tau_{\mathbb{Y}}.$$

Since $\mathscr{B}(\mathbb{Y})$ is generated by $\tau_{\mathbb{Y}}$, it follows from Proposition 3.3 that f is measurable. ∎

Comment: Functions into topological spaces are of particular importance, and specifically real-valued functions. Let $(\mathbb{X}, \Sigma)$ be a measurable space and let $f : \mathbb{X} \to \mathbb{R}$. Unless otherwise specified, we say that f is measurable if it is $(\Sigma, \mathscr{B}(\mathbb{R}))$-measurable. A function $f : \mathbb{R} \to \mathbb{R}$ is called **Borel-measurable** if it is $(\mathscr{B}(\mathbb{R}), \mathscr{B}(\mathbb{R}))$-measurable and it is called **Lebesgue-measurable** if it is $(\mathcal{L}, \mathscr{B}(\mathbb{R}))$-measurable. We use the same terminology when the range is the field of complex numbers $\mathbb{C}$.

Comment: The composition of measurable maps between measure spaces is measurable: indeed if

$$(\mathbb{X}, \Sigma_{\mathbb{X}}) \xrightarrow{f} (\mathbb{Y}, \Sigma_{\mathbb{Y}}) \xrightarrow{g} (\mathbb{Z}, \Sigma_{\mathbb{Z}})$$

are measurable, then

$$(g \circ f)^{-1}(\Sigma_{\mathbb{Z}}) = f^{-1}(g^{-1}(\Sigma_{\mathbb{Z}})) \subset f^{-1}(\Sigma_{\mathbb{Y}}) \subset \Sigma_{\mathbb{X}}.$$

In particular, the composition of Borel-measurable maps

$$(\mathbb{R}, \mathscr{B}(\mathbb{R})) \xrightarrow{f} (\mathbb{R}, \mathscr{B}(\mathbb{R}))) \xrightarrow{g} (\mathbb{R}, \mathscr{B}(\mathbb{R}))$$

is Borel-measurable. However, the composition of Lebesgue-measurable maps

$$(\mathcal{L}, \mathscr{B}(\mathbb{R})) \xrightarrow{f} (\mathbb{R}, \mathscr{B}(\mathbb{R}))) \qquad \text{and} \qquad (\mathcal{L}, \mathscr{B}(\mathbb{R})) \xrightarrow{g} (\mathbb{R}, \mathscr{B}(\mathbb{R})))$$

is not necessarily Lebesgue-measurable.

Comment: We will also consider complex-valued functions. Topologically (though not algebraically), the complex plane $\mathbb{C}$ is homeomorphic to $\mathbb{R}^2$. Thus,

$$\mathscr{B}(\mathbb{C}) \simeq \mathscr{B}(\mathbb{R}^2) \simeq \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}),$$

where the product σ-algebra was defined in Section 2.1.3.

By the definition of the product σ-algebra, and since the projection maps $\mathbb{C} \to \mathbb{R}$ are the real and the imaginal part,

$$\mathscr{B}(\mathbb{C}) = \sigma\left(\{\mathrm{Re}^{-1}(E) \,:\, E \in \mathscr{B}(\mathbb{R})\} \cup \{\mathrm{Im}^{-1}(F) \,:\, F \in \mathscr{B}(\mathbb{R})\}\right).$$

By Proposition 3.3, f is measurable if and only if for every $E, F \in \mathscr{B}(\mathbb{R})$,

$$f^{-1}(\mathrm{Re}^{-1}(E)) \in \Sigma_{\mathbb{X}} \qquad \text{and} \qquad f^{-1}(\mathrm{Im}^{-1}(F)) \in \Sigma_{\mathbb{X}},$$

namely, if and only if

$$(\mathrm{Re}\, f)^{-1}(E) \in \Sigma_{\mathbb{X}} \qquad \text{and} \qquad (\mathrm{Im}\, f)^{-1}(F) \in \Sigma_{\mathbb{X}},$$

i.e., if and only if

$$\mathrm{Re}\, f \quad \text{and} \quad \mathrm{Im}\, f \quad \text{are measurable.}$$

Let $(\mathbb{Y}, \Sigma_{\mathbb{Y}})$ be a measurable space and let $f : \mathbb{X} \to \mathbb{Y}$, where $\mathbb{X}$ is some non-empty set. We saw that if we endow $\mathbb{X}$ with the maximal σ-algebra $\mathscr{P}(\mathbb{X})$, then f is measurable.

Definition 3.5 Let $(\mathbb{Y}, \Sigma_{\mathbb{Y}})$ be a measurable space and let $f : \mathbb{X} \to \mathbb{Y}$. The ***$\sigma$-algebra generated by*** *f is the intersection of all σ-algebras on $\mathbb{X}$ with respect to which f is measurable.*

It is easy to see that the σ-algebra generated by f is $f^{-1}(\Sigma_{\mathbb{Y}})$.

Proposition 3.6 Let $(\mathbb{X}, \Sigma)$ be a measurable space and let $f : \mathbb{X} \to \mathbb{R}$. Then, the following are equivalent:

1. *f is measurable.*
2. *$f^{-1}((a, \infty)) \in \Sigma$ for all $a \in \mathbb{R}$.*
3. *$f^{-1}([a, \infty)) \in \Sigma$ for all $a \in \mathbb{R}$.*
4. *$f^{-1}((-\infty, a)) \in \Sigma$ for all $a \in \mathbb{R}$.*
5. *$f^{-1}((-\infty, a]) \in \Sigma$ for all $a \in \mathbb{R}$.*

Proof: This follows from Proposition 3.3 and the fact that each of these sets generates $\mathscr{B}(\mathbb{R})$ (Proposition 2.9). ■ —13h(2017)—

We next verify that the notion of measurability of functions pieces together with basic algebraic operations on functions:

Proposition 3.7 *If $f, g : (\mathbb{X}, \Sigma) \to \mathbb{C}$ are measurable, then so are $f+g$ and fg.*

Proof: Define $F : \mathbb{X} \to \mathbb{C} \times \mathbb{C}$ by

$$F(x) = (f(x), g(x)),$$

and $\phi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by

$$\phi(z, w) = z + w.$$

Since $\mathscr{B}(\mathbb{C} \times \mathbb{C}) = \mathscr{B}(\mathbb{C}) \otimes \mathscr{B}(\mathbb{C})$, it is generated by the sets

$$\{A \times \mathbb{C} \,:\, A \in \mathscr{B}(\mathbb{C})\} \qquad \text{and} \qquad \{\mathbb{C} \times B \,:\, B \in \mathscr{B}(\mathbb{C})\}.$$

Now,

$$F^{-1}(\{A \times \mathbb{C} \,:\, A \in \mathscr{B}(\mathbb{C})\}) = \{f^{-1}(A) \,:\, A \in \mathscr{B}(\mathbb{C})\}) \subset \Sigma$$

and

$$F^{-1}(\{\mathbb{C} \times B \,:\, B \in \mathscr{B}(\mathbb{C})\}) = \{g^{-1}(B) \,:\, B \in \mathscr{B}(\mathbb{C})\}) \subset \Sigma,$$

proving that F is measurable. Likewise, ϕ is measurable because it is continuous. It follows that $\phi \circ F$, given by

$$\phi \circ F(x) = f(x) + g(x)$$

is measurable. The second part is proved similarly with $\phi(z, w) = zw$. ■

Proposition 3.8 *Let $f_n : \mathbb{X} \to \mathbb{R}$ be a sequence of measurable functions. Then, the functions*

$$g_1(x) = \sup_n f_n(x)$$
$$g_2(x) = \inf_n f_n(x)$$
$$g_3(x) = \limsup_n f_n(x)$$
$$g_4(x) = \liminf_n f_n(x)$$

are measurable (here we use the topology of the extended real line, to allow for infinite limits).

Comment: The extended real line $\bar{\mathbb{R}}$ is obtained from the real number system $\mathbb{R}$ by adding two elements: $-\infty$ and ∞. Topologically, a set U is a neighborhood of ∞ if it contains a set (a, ∞), and analogously for the neighborhoods of $-\infty$ (this topology is an **order topology** (**טופולוגיית סדר**) applicable to every totally-ordered set). With this topology, the limits ∞ and $-\infty$ reduce to the standard topological definitions of limits.

Proof: Start with g_1. For every $a \in \mathbb{R}$,

$$\begin{aligned} g_1^{-1}((a,\infty]) &= \{x \in \mathbb{X} \ : \ \sup_n f_n(x) > a\} \\ &= \{x \in \mathbb{X} \ : \ \exists n, \ \ f_n(x) > a\} \\ &= \bigcup_{n=1}^{\infty} f_n^{-1}((a,\infty]) \in \Sigma \end{aligned}$$

(the supremum at x is greater than a if and only if there exists an n for which $f_n(x) > a$). This implies that g_1 is measurable. We proceed similarly for g_2. As for g_3 and g_4,

$$g_3(x) = \inf_n \sup_{k\ge n} f_k(x) \qquad \text{and} \qquad g_4(x) = \sup_n \inf_{k\ge n} f_k(x)$$

which are measurable by the first two items. ■

Corollary 3.9 *If f and g are measurable then so are*

$$\max(f,g) \qquad \textit{and} \qquad \min(f,g).$$

Proof: Immediate from the previous proposition, taking $f_{2n} = f$ and $f_{2n+1} = g$. ■

Corollary 3.10 *Let $f_n : \mathbb{X} \to \mathbb{C}$ be a sequence of complex-valued measurable functions. If*

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists for all $x \in \mathbb{X}$, then f is measurable.

Definition 3.11 *Given a function $f : \mathbb{X} \to \mathbb{R}$, we denote its positive and negative parts by*

$$f^+ = \max(f, 0) \qquad \textit{and} \qquad f^- = \max(-f, 0).$$

If f is measurable then by Corollary 3.9 both f^+ and f^- are measurable (every constant function is measurable, see Ex. 3.1). Note also that

$$f = f^+ - f^- \qquad \text{and} \qquad |f| = f^+ + f^-,$$

hence the latter is measurable.

✎ *Exercise 3.2* Let $(\mathbb{X}, \Sigma)$ be a measurable space, and let $\mathcal{D} \subset \mathbb{R}$ be a dense set. Let $f : \mathbb{X} \to \bar{\mathbb{R}}$ satisfy

$$\{x : f(x) > c\} \in \Sigma$$

for all $c \in \mathcal{D}$. Prove that f is measurable.

✎ *Exercise 3.3* Let $(\mathbb{X}, \Sigma)$ be a measurable space, and let $A, B \in \Sigma$. Prove that $f : A \cup B \to \mathbb{R}$ is measurable if and only if its restrictions $f|_A$ and $f|_B$ are measurable.

✎ *Exercise 3.4* Show that the following functions $\mathbb{R} \to \mathbb{R}$ are Borel-measurable:

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} x & x \in \mathbb{Q} \\ -x & x \notin \mathbb{Q} \end{cases}.$$

✎ *Exercise 3.5* Show that every monotone function $f : \mathbb{R} \to \mathbb{R}$ is measurable.

✎ *Exercise 3.6* Let $(\mathbb{X}, \Sigma)$ be a measurable space, and let $f_n : \mathbb{X} \to \mathbb{R}$ be a sequence of measurable functions. Prove that the set of points in $\mathbb{X}$ on which f_n converges is measurable.

3.2 Simple functions

Definition 3.12 *Let $(\mathbb{X}, \Sigma)$ be a measurable space. For $A \subset \mathbb{X}$, its* ***characteristic function*** *(פונקציה מציינת) is defined by*

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

Proposition 3.13 *χ_A is a measurable function if and only if A is a measurable set.*

Proof: This is immediate from the definition, as for every $B \in \mathscr{B}(\mathbb{R})$,

$$\chi_A^{-1}(B) = \begin{cases} A & 1 \in B \quad \text{and} \quad 0 \notin B \\ A^c & 0 \in B \quad \text{and} \quad 1 \notin B \\ \mathbb{X} & 0, 1 \in B \\ \varnothing & 0, 1 \notin B. \end{cases}$$

■

Definition 3.14 *Let $(\mathbb{X}, \Sigma)$ be a measurable space. A **simple function** (פונקציה פשוטה) on $\mathbb{X}$ is a finite linear combination of characteristic functions of measurable sets with complex (or real) coefficients. That is,*

$$f(x) = \sum_{j=1}^{n} z_j \chi_{A_j}(x), \qquad A_j \in \Sigma.$$

*A simple function is in **standard representation** (הצגה תקנית) if the A_j are disjoint. We denote the algebra of simple functions by $\mathrm{SF}(\mathbb{X}, \Sigma)$.*

Simple functions will be used repeatedly to approximate measurable functions:

Theorem 3.15 (Approximation of non-negative functions by simple functions) *Let $(\mathbb{X}, \Sigma)$ be a measurable space. If $f : \mathbb{X} \to [0, \infty]$ is measurable, then there exists a sequence of simple functions, $\phi_i \in \mathrm{SF}(\mathbb{X}, \Sigma)$,*

$$0 \le \phi_1 \le \phi_2 \le \cdots \le f,$$

converging to f pointwise, and uniformly on any set in which f is bounded.

Proof: For every $n \in \{0\} \cup \mathbb{N}$ and $0 \le k \le 2^{2n} - 1$, let

$$E_n^k = f^{-1}((k2^{-n}, (k+1)2^{-n})) \qquad \text{and} \qquad F_n = f^{-1}((2^n, \infty])$$

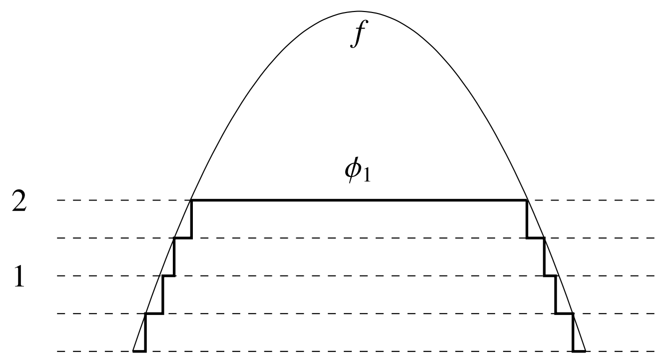
(note that $k2^{-n}$ ranges from 0 to $2^n - 2^{-n}$ in steps of 2^{-n}). Define

$$\phi_n(x) = \sum_{k=0}^{2^{2n}-1} k2^{-n}\chi_{E_n^k}(x) + 2^n\chi_{F_n}(x).$$

It is easy to set that $\phi_n \le f$ and that at all points where $f(x) \le 2^n$,

$$f(x) - \phi_n(x) \le 2^{-n},$$

hence the uniform convergence on sets where f is bounded.



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Proposition 3.16 Let $(\mathbb{X}, \Sigma, \mu)$ be a complete measure space. Then,

1. *If f is measurable and $g = f$ μ-a.e., then g is measurable.*
2. *If f_n are measurable functions converging to f μ-a.e., then f is measurable.*

Proof: For the first part, let

$$A = \{x \in \mathbb{X} \,:\, f(x) = g(x)\}.$$

It is given that $\mu(A^c) = 0$ (hence A is measurable). Let $B \in \mathscr{B}(\mathbb{R})$. Then,

$$\begin{aligned} g^{-1}(B) &= (g^{-1}(B) \cap A) \cup (g^{-1}(B) \cap A^c) \\ &= \underbrace{(f^{-1}(B) \cap A)}_{\in\Sigma} \cup (g^{-1}(B) \cap A^c). \end{aligned}$$

Since μ is complete, then $g^{-1}(B) \cap A^c \subset A^c$ is measurable, hence $g^{-1}(B)$ is measurable, proving that g is measurable.

For the second part, let

$$A = \{x \in \mathbb{X} \,:\, f_n(x) \to f(x)\}.$$

Since A^c has measure zero, A is measurable, and so are the functions $f_n|_A$ (see Ex. 3.3). Since moreover $f_n|_A \to f|_A$, it follows that $f|_A$ is measurable (Corollary 3.10). Define

$$\tilde{f}(x) = \begin{cases} f(x) & x \in A \\ 0 & x \notin A. \end{cases}$$

Then, $\tilde{f}$ is measurable (once again see Ex. 3.3) and equals f μ-a.e. It follows from the first part that f is measurable. ∎

✎ *Exercise 3.7* Let $(\mathbb{X}, \Sigma, \mu)$ be a measure space and let $(\mathbb{X}, \Sigma', \mu')$ be its completion. Prove that for every $f : \mathbb{X} \to \mathbb{R}$ which is μ'-measurable, there exists an $\tilde{f} : \mathbb{X} \to \mathbb{R}$ which is μ-measurable and equals f μ-a.e.

3.3 Integration of non-negative functions

Having a notion of (real- or complex-valued) measurable functions, we proceed to define their integral over a measure space. We proceed in stages, starting with real-valued function that assume non-negative values.

Definition 3.17 *Let $(\mathbb{X}, \Sigma)$ be a measure space. We denote by $L^+(\mathbb{X}, \Sigma)$ the space of all measurable functions $\mathbb{X} \to [0, \infty]$; we denote by $\mathrm{SF}^+(\mathbb{X}, \Sigma)$ the space of non-negative simple functions.*

Definition 3.18 *Let $\phi \in \mathrm{SF}^+(\mathbb{X}, \Sigma)$ be given by*

$$\phi(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x).$$

The ***integral*** *of ϕ with respect to μ is defined by*

$$\int_{\mathbb{X}} \phi \, d\mu = \sum_{j=1}^{n} a_j \mu(E_j). \tag{3.2}$$

If $A \subset \mathbb{X}$ is a measurable set, we define

$$\int_A \phi \, d\mu = \int_{\mathbb{X}} \chi_A \phi \, d\mu,$$

where we use the fact that

$$\chi_A \phi(x) = \sum_{j=1}^n a_j \chi_{E_j \cap A}(x)$$

is a simple function as well, hence

$$\int_A \phi \, d\mu = \sum_{j=1}^n a_j \mu(E_j \cap A).$$

Comment: Other standard notations for the integral are

$$\int_{\mathbb{X}} \phi(x) \, d\mu(x) \qquad \text{and} \qquad \int_{\mathbb{X}} \phi(x) \, \mu(dx).$$

Note that these are just notations, which like for the Riemann integral may provide additional insight into the definition.

Example: Consider the segment $[0,1]$ with the σ-algebra of Borel sets. The Dirichlet function

$$D = 1 \cdot \chi_{\mathbb{Q}^c} + 0 \cdot \chi_{\mathbb{Q}}$$

is a simple function (yes!) and its integral is

$$\int_{[0,1]} D \, dm = 1 \cdot m(\mathbb{Q}^c) = 1.$$

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The next proposition shows that integrals of non-negative simple functions behave as we would like them to behave:

Proposition 3.19 Let $f, g \in \mathrm{SF}^+(\mathbb{X}, \Sigma)$ and let $c > 0$. Then,

(a) $\int_{\mathbb{X}} cf \, d\mu = c \int_{\mathbb{X}} f \, d\mu$.

(b) $\int_{\mathbb{X}} (f+g) \, d\mu = \int_{\mathbb{X}} f \, d\mu + \int_{\mathbb{X}} g \, d\mu$.

(c) *If* $f \le g$ *then* $\int_{\mathbb{X}} f \, d\mu \le \int_{\mathbb{X}} g \, d\mu$.

(d) *The map* $A \mapsto \int_A f \, d\mu$ *is a measure on* Σ.

Proof: Item (a) follows directly from the definition. For Item (b), let

$$f = \sum_{j=1}^{n} a_j \chi_{E_j} \qquad \text{and} \qquad g = \sum_{k=1}^{m} b_k \chi_{F_k}.$$

be in standard notation, i.e., $\{E_j\}$ and $\{F_k\}$ are disjoint. Then,

$$f + g = \sum_{j=1}^{n} \sum_{k=1}^{m} (a_j + b_k) \chi_{E_j \cap F_k}.$$

Hence,

$$\begin{aligned}
\int_{\mathbb{X}} (f+g)\, d\mu &= \sum_{j=1}^{n} \sum_{k=1}^{m} (a_j + b_k) \mu(E_j \cap F_k) \\
&= \sum_{j=1}^{n} a_j \sum_{k=1}^{m} \mu(E_j \cap F_k) + \sum_{k=1}^{m} b_k \sum_{j=1}^{n} \mu(E_j \cap F_k) \\
&= \sum_{j=1}^{n} a_j \mu(\sqcup_{k=1}^{m} (E_j \cap F_k)) + \sum_{k=1}^{m} b_k \mu(\sqcup_{j=1}^{n} (E_j \cap F_k)) \\
&= \sum_{j=1}^{n} a_j \mu(E_j) + \sum_{k=1}^{m} b_k \mu(F_k) \\
&= \int_{\mathbb{X}} f\, d\mu + \int_{\mathbb{X}} g\, d\mu.
\end{aligned}$$

For Item (c) we write

$$f = \sum_{j=1}^{n} \sum_{k=1}^{m} a_j \chi_{E_j \cap F_k} \qquad \text{and} \qquad g = \sum_{j=1}^{n} \sum_{k=1}^{m} b_k \chi_{E_j \cap F_k},$$

and note that $a_j \leq b_k$ whenever $E_j \cap F_k \neq \varnothing$. Hence

$$\int_{\mathbb{X}} f\, d\mu = \sum_{j=1}^{n} \sum_{k=1}^{m} a_j \mu(E_j \cap F_k) \leq \sum_{j=1}^{n} \sum_{k=1}^{m} b_k \mu(E_j \cap F_k) = \int_{\mathbb{X}} g\, d\mu.$$

Finally, for Item (d), let

$$\nu(A) = \int_A f\, d\mu.$$

Clearly, $\nu(\varnothing) = 0$. Let (A_n) be a sequence of disjoint measurable sets. Then,

$$\begin{aligned}
\nu\left(\coprod_{n=1}^{\infty} A_n\right) &= \int_{\coprod_{n=1}^{\infty} A_n} f\,d\mu \\
&= \int_{\mathbb{X}} \chi_{\coprod_{n=1}^{\infty} A_n} f\,d\mu \\
&= \sum_{j=1}^{n} a_j \mu\left(E_j \cap \coprod_{n=1}^{\infty} A_n\right) \\
&= \sum_{j=1}^{n} a_j \sum_{n=1}^{\infty} \mu(E_j \cap A_n) \\
&= \sum_{n=1}^{\infty} \sum_{j=1}^{n} a_j \mu(E_j \cap A_n) \\
&= \sum_{n=1}^{\infty} \int_{A_n} f\,d\mu \\
&= \sum_{n=1}^{\infty} \nu(A_n).
\end{aligned}$$

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Having a definition for the integral of simple functions, we proceed to extend the definition for any function in $L^+(\mathbb{X}, \Sigma)$:

Definition 3.20 Let $f \in L^+(\mathbb{X}, \Sigma)$. Then,

$$\int_{\mathbb{X}} f\,d\mu = \sup\left\{\int_{\mathbb{X}} \phi\,d\mu \;:\; \phi \le f, \quad \phi \in \mathrm{SF}^+(\mathbb{X}, \Sigma)\right\}. \tag{3.3}$$

Proposition 3.21 For $f \in \mathrm{SF}^+(\mathbb{X}, \Sigma)$, the definitions (3.2) and (3.3) coincide. Moreover, for $f \le g \in L^+(\mathbb{X}, \Sigma)$ and $c > 0$,

$$\int_{\mathbb{X}} f\,d\mu \le \int_{\mathbb{X}} g\,d\mu \qquad \text{and} \qquad \int_{\mathbb{X}} cf\,d\mu = c\int_{\mathbb{X}} f\,d\mu.$$

Proof: That the two definitions coincide is obvious, as if $f \in \mathrm{SF}^+(\mathbb{X}, \Sigma)$,

$$\int_{\mathbb{X}} f\,d\mu = \max\left\{\int_{\mathbb{X}} \phi\,d\mu \;:\; \phi \le f, \quad \phi \in \mathrm{SF}^+(\mathbb{X}, \Sigma)\right\}.$$

If $f \le g \in L^+(\mathbb{X},\Sigma)$ then

$$\left\{ \int_{\mathbb{X}} \phi\, d\mu \ :\ \phi \le f, \quad \phi \in \mathrm{SF}^+(\mathbb{X},\Sigma) \right\} \subset \left\{ \int_{\mathbb{X}} \phi\, d\mu \ :\ \phi \le g, \quad \phi \in \mathrm{SF}^+(\mathbb{X},\Sigma) \right\},$$

hence the inequality between the supremums. Finally, for $c > 0$

$$\begin{aligned} \left\{ \int_{\mathbb{X}} \phi\, d\mu \ :\ \phi \le cf, \quad \phi \in \mathrm{SF}^+(\mathbb{X},\Sigma) \right\} &= \left\{ \int_{\mathbb{X}} c\phi\, d\mu \ :\ c\phi \le cf, \quad \phi \in \mathrm{SF}^+(\mathbb{X},\Sigma) \right\} \\ &= c \left\{ \int_{\mathbb{X}} \phi\, d\mu \ :\ \phi \le f, \quad \phi \in \mathrm{SF}^+(\mathbb{X},\Sigma) \right\}. \end{aligned}$$

■

The following theorem is a central pillar in integration theory:

Theorem 3.22 (Monotone Convergence (התכנסות מונוטונית)) Let $f_n \in L^+(\mathbb{X},\Sigma)$ be monotonically increasing, $f_n \le f_{n+1}$. Let

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Then, $f \in L^+(\mathbb{X},\Sigma)$ and

$$\int_{\mathbb{X}} f\, d\mu = \lim_{n\to\infty} \int_{\mathbb{X}} f_n\, d\mu.$$

—15h(2017)—

Proof: Since (f_n) is increasing, it converges everywhere (possibly assuming infinite values); limits of measurable functions are measurable, hence $f \in L^+(\mathbb{X},\Sigma)$. Moreover, since $f_n \le f$,

$$\int_{\mathbb{X}} f_n\, d\mu \le \int_{\mathbb{X}} f\, d\mu,$$

hence

$$\lim_{n\to\infty} \int_{\mathbb{X}} f_n\, d\mu \le \int_{\mathbb{X}} f\, d\mu.$$

For the reverse inequality, let $0 < \alpha < 1$ and let $\phi \le f$ be a non-negative simple function. Define

$$E_n = \{x \in \mathbb{X} \ :\ f_n(x) \ge \alpha\phi(x)\}.$$

This sequence of sets is increasing (by the monotonicity of f_n), measurable (because $f_n - \alpha\phi$ is measurable) and its union is $\mathbb{X}$ (because for every x, $f_n(x)$ is eventually larger than $\alpha\phi(x) \le \alpha f(x)$). Now,

$$\int_{\mathbb{X}} f_n \, d\mu \ge \int_{E_n} f_n \, d\mu \ge \alpha \int_{E_n} \phi \, d\mu,$$

where in the last step we used the defining property of points in E_n. Define

$$\nu(A) = \int_A \phi \, d\mu,$$

i.e.,

$$\int_{\mathbb{X}} f_n \, d\mu \ge \alpha \, \nu(E_n).$$

We have seen that ν is a measure. By the lower-semicontinuity of measures,

$$\lim_{n\to\infty} \int_{\mathbb{X}} f_n \, d\mu \ge \alpha \lim_{n\to\infty} \nu(E_n) = \alpha \, \nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \alpha \, \nu(\mathbb{X}) = \alpha \int_{\mathbb{X}} \phi \, d\mu.$$

Letting $\alpha \to 1$ and taking the supremum over all $\phi \le f$ we obtain that

$$\lim_{n\to\infty} \int_{\mathbb{X}} f_n \, d\mu \ge \int_{\mathbb{X}} f \, d\mu,$$

which completes the proof. ∎

The Monotone Convergence Theorem has a very practical implication. The definition of the integral of $f \in L^+(\mathbb{X}, \Sigma)$ involves a supremum over a huge set of functions. By Monotone Convergence, it can be obtained as a limit over integrals of simple functions increasing to f (and those always exist by Theorem 3.15). We will now derive a number of almost immediate consequences of Monotone Convergence.

Proposition 3.23 *If $f, g \in L^+(\mathbb{X}, \Sigma)$, then*

$$\int_{\mathbb{X}} (f + g) \, d\mu = \int_{\mathbb{X}} f \, d\mu + \int_{\mathbb{X}} g \, d\mu.$$

Proof: Let $\phi_n, \psi_n \in \mathrm{SF}^+(\mathbb{X}, \Sigma)$ be increasing to f and g, then $\phi_n + \psi_n$ increases to $f + g$, and by Monotone Convergence,

$$\begin{aligned}\int_{\mathbb{X}} (f+g)\, d\mu &= \lim_{n\to\infty} \int_{\mathbb{X}} (\phi_n + \psi_n)\, d\mu \\ &= \lim_{n\to\infty} \left(\int_{\mathbb{X}} \phi_n\, d\mu + \int_{\mathbb{X}} \psi_n\, d\mu \right) \\ &= \int_{\mathbb{X}} f\, d\mu + \int_{\mathbb{X}} g\, d\mu.\end{aligned}$$

■

Proposition 3.24 *Let $f_n \in L^+(\mathbb{X}, \Sigma)$. Then,*

$$\int_{\mathbb{X}} \sum_{n=1}^{\infty} f_n\, d\mu = \sum_{n=1}^{\infty} \int_{\mathbb{X}} f_n\, d\mu.$$

(Verify that both sides are indeed well defined.)

Proof: By induction,

$$\int_{\mathbb{X}} \sum_{n=1}^{N} f_n\, d\mu = \sum_{n=1}^{N} \int_{\mathbb{X}} f_n\, d\mu.$$

Letting $N \to \infty$ and applying the Monotone Convergence Theorem, we obtain the desired result. ■

✎ *Exercise 3.8* Let $(\mathbb{X}, \Sigma, \mu)$ be a σ-finite measure space. Let $f : \mathbb{X} \to [0, \infty)$ be measurable. Show that

$$\int_{\mathbb{X}} f\, d\mu = \int_{[0,\infty)} \mu(\{x \,:\, f(x) > t\})\, dm(t).$$

Hint: establish the identity first for indicator functions, then for functions in $\mathrm{SF}^+(\mathbb{X}, \Sigma)$, and finally for functions in $L^+(\mathbb{X}, \Sigma)$.

Proposition 3.25 *If $f \in L^+(\mathbb{X}, \Sigma)$, then*

$$\int_{\mathbb{X}} f\, d\mu = 0$$

if and only if $f = 0$ μ-a.e.,

Proof: If ϕ is simple then, by definition, its integral is zero if and only if it equals zero μ-a.e. Suppose that $f \in L^+(\mathbb{X},\Sigma)$ and equals zero μ-a.e. Then, every simple $0 \le \phi \le f$ is zero μ-a.e., i.e.,

$$\int_{\mathbb{X}} f\, d\mu = \sup_{\phi \le f} \int_{\mathbb{X}} \phi\, d\mu = 0.$$

Conversely, suppose that $f \in L^+(\mathbb{X},\Sigma)$ and

$$\int_{\mathbb{X}} f\, d\mu = 0.$$

Then,

$$\{x \in \mathbb{X} \,:\, f(x) > 0\} = \bigcup_{n=1}^{\infty} \{x \in \mathbb{X} \,:\, f(x) > 1/n\} \equiv \bigcup_{n=1}^{\infty} E_n,$$

with (E_n) increasing. By the lower-semicontinuity of μ

$$\mu(\{x \in \mathbb{X} \,:\, f(x) > 0\}) = \lim_{n\to\infty} \mu(E_n).$$

If the left-hand side equals $c > 0$, then there exists an n for which $\mu(E_n) > c/2$. Then,

$$\int_{\mathbb{X}} f\, d\mu \ge \int_{E_n} f\, d\mu \ge \frac{1}{n} \int_{E_n} d\mu > \frac{c}{2n} > 0,$$

which is a contradiction. ∎

Corollary 3.26 If $f_n \in L^+(\mathbb{X},\Sigma)$ increases to f μ-a.e., then

$$\int_{\mathbb{X}} f\, d\mu = \lim_{n\to\infty} \int_{\mathbb{X}} f_n\, d\mu.$$

Proof: Let E be the set on which f_n increases to f. Then,

$$f - f\chi_E = 0 \quad \mu\text{-a.e.},$$

and

$$f_n - f_n\chi_E = 0 \quad \mu\text{-a.e.}$$

By the previous proposition,

$$\int_{\mathbb{X}} (f - f\chi_E)\, d\mu = 0 = \lim_{n\to\infty} \int_{\mathbb{X}} (f_n - f_n\chi_E)\, d\mu,$$

i.e.,

$$\int_{\mathbb{X}} f\chi_E \, d\mu = \int_{\mathbb{X}} f \, d\mu \qquad \text{and} \qquad \int_{\mathbb{X}} f_n\chi_E \, d\mu = \int_{\mathbb{X}} f_n \, d\mu.$$

By Monotone Convergence, since $f_n\chi_E$ increases to $f\chi_E$ *everywhere*,

$$\lim_{n\to\infty} \int_{\mathbb{X}} f_n\chi_E \, d\mu = \int_{\mathbb{X}} f\chi_E \, d\mu.$$

To conclude:

$$\int_{\mathbb{X}} f \, d\mu = \int_{\mathbb{X}} f\chi_E \, d\mu = \lim_{n\to\infty} \int_{\mathbb{X}} f_n\chi_E \, d\mu = \lim_{n\to\infty} \int_{\mathbb{X}} f_n \, d\mu.$$

■

✎ *Exercise 3.9* Let $(\mathbb{R}, \mathscr{B}(\mathbb{R}), m)$ be the standard Borel measure space, and let $f \in L^+(\mathbb{R})$. Suppose that

$$F(x) = \sum_{n=1}^{\infty} f(x+n)$$

has finite integral. Prove that $f = 0$ a.e.

✎ *Exercise 3.10* Let $(\mathbb{X}, \Sigma, \mu)$ be a measure space and let $f \in L^+(\mathbb{X}, \Sigma)$ have finite integral. Show that there exists for every $\varepsilon > 0$ a $\delta > 0$, such that $\mu(A) < \delta$ implies

$$\int_A f \, d\mu < \varepsilon.$$

—16h(2017)—

TA material 3.1 Show what may go wrong when the convergence is not monotone.

In the Monotone Convergence Theorem, the sequence of functions is increasing, hence bounded from above by a limit f. For an arbitrary sequence in $L^+(\mathbb{X}, \Sigma)$, we have the following:

Proposition 3.27 (Fatou's lemma) Let $f_n \in L^+(\mathbb{X}, \Sigma)$ be an arbitrary sequence. Then.

$$\int_{\mathbb{X}} \liminf_{n\to\infty} f_n \, d\mu \le \liminf_{n\to\infty} \int_{\mathbb{X}} f_n \, d\mu.$$

Comment: In particular, if $f_n \to f$, then

$$\int_{\mathbb{X}} f\, d\mu \le \liminf_{n\to\infty} \int_{\mathbb{X}} f_n\, d\mu.$$

Proof: First note that the left-hand side is well-defined as the inferior-limit of a measurable sequence is measurable. Since

$$\liminf_{n\to\infty} f_n = \lim_{n\to\infty} \inf_{k\ge n} f_k,$$

and $\inf_{k\ge n} f_k$ is increasing, it follows from Monotone Convergence that for every n,

$$\int_{\mathbb{X}} \liminf_{n\to\infty} f_n\, d\mu = \lim_{n\to\infty} \int_{\mathbb{X}} \inf_{k\ge n} f_k\, d\mu.$$

Finally, since for every n and $\ell \ge n$

$$\int_{\mathbb{X}} \inf_{k\ge n} f_k\, d\mu \le \int_{\mathbb{X}} f_\ell\, d\mu,$$

it follows that

$$\int_{\mathbb{X}} \inf_{k\ge n} f_k\, d\mu \le \inf_{\ell\ge n} \int_{\mathbb{X}} f_\ell\, d\mu,$$

hence

$$\int_{\mathbb{X}} \liminf_{n\to\infty} f_n\, d\mu \le \lim_{n\to\infty} \inf_{\ell\ge n} \int_{\mathbb{X}} f_\ell\, d\mu = \liminf_{n\to\infty} \int_{\mathbb{X}} f_n\, d\mu.$$

■

Corollary 3.28 If $f_n, f \in L^+(\mathbb{X}, \Sigma)$ and $f_n \to f$ μ-a.e., then

$$\int_{\mathbb{X}} f\, d\mu \le \liminf_{n\to\infty} \int_{\mathbb{X}} f_n\, d\mu.$$

Proof: If $f_n \to f$ everywhere then this follows from Fatou's lemma. Since we can modify f on a set of measure zero without affecting the integral we obtain the desired result. Specifically, we set

$$E = \left\{ x \,:\, \lim_{n\to\infty} f_n(x) = f(x) \right\}.$$

Then $f_n\chi_E$ converges to $f\chi_E$ everywhere, and it follows from Fatou's lemma that

$$\int_{\mathbb{X}} f\chi_E\, d\mu \le \liminf_{n\to\infty} \int_{\mathbb{X}} f_n\chi_E\, d\mu.$$

Since $f\chi_E = f$ and $f_n\chi_E = f_n$ μ-a.e., we obtain the desired result. ■

Proposition 3.29 If $f \in L^+(\mathbb{X},\Sigma)$ and $\int_{\mathbb{X}} f\, d\mu < \infty$, then

$$\mu(\{x : f(x) = \infty\}) = 0,$$

and the set

$$\{x : f(x) > 0\}$$

is σ-finite.

Proof: Consider the set

$$A = \{x : f(x) = \infty\},$$

which is measurable since

$$A = \bigcap_{n=1}^{\infty} f^{-1}((n,\infty]).$$

Suppose A had finite measure, $\mu(A) = c > 0$. Consider the sequence of simple functions,

$$\phi_n(x) = \begin{cases} n & x \in A \\ 0 & x \notin A. \end{cases}$$

Then, $\phi_n \le f$ for all n. Since

$$\int_{\mathbb{X}} \phi_n\, d\mu = cn$$

it follows that

$$\int_{\mathbb{X}} f\, d\mu \ge \sup_{n\in\mathbb{N}} \int_{\mathbb{X}} \phi_n\, d\mu = \infty,$$

which is a contradiction.

For the second part, let

$$B_n = \left\{x : f(x) \ge \frac{1}{n}\right\}.$$

Then,

$$\int_{\mathbb{X}} f\,d\mu \geq \int_{B_n} f\,d\mu \geq \frac{\mu(B_n)}{n},$$

i.e.,

$$\mu(B_n) \leq n \int_{\mathbb{X}} f\,d\mu < \infty.$$

Since

$$\{x : f(x) > 0\} = \bigcup_{n=1}^{\infty} B_n,$$

the left-hand side is σ-finite. ■

Proposition 3.30 (Borel-Cantelli) Let A_n be measurable sets satisfying

$$\sum_{n=1}^{\infty} \mu(A_n) < \infty.$$

Then,

$$\mu\left(\limsup_{n\to\infty} A_n\right) = \mu\left(\{x : x \in A_n \textit{ for infinitely many } n\text{'s}\}\right) = 0.$$

Proof: Take $f_n = \chi_{A_n} \in L^+(\mathbb{X}, \Sigma)$ and apply Proposition 3.24. Then,

$$\int_{\mathbb{X}} \sum_{n=1}^{\infty} \chi_{A_n}\,d\mu = \sum_{n=1}^{\infty} \int_{\mathbb{X}} \chi_{A_n}\,d\mu = \sum_{n=1}^{\infty} \mu(A_n) < \infty.$$

It follows from the previous proposition that

$$\begin{aligned} 0 &= \mu\left(\left\{x : \sum_{n=1}^{\infty} \chi_{A_n}(x) = \infty\right\}\right) \\ &= \mu\left(\{x : x \in A_n \text{ for infinitely many } n\text{'s}\}\right). \end{aligned}$$

—16h(2018)— ■

3.4 Integration of complex functions

Having defined integrals for *non-negative*, measurable real-valued functions, we proceed to define the integral of general measurable (real- and complex-valued) functions.

Definition 3.31 Let $(\mathbb{X}, \Sigma, \mu)$ be a measure space and let $f : \mathbb{X} \to \mathbb{R}$ be measurable. If either

$$\int_{\mathbb{X}} f^+ \, d\mu < \infty \qquad \text{or} \qquad \int_{\mathbb{X}} f^- \, d\mu < \infty,$$

then

$$\int_{\mathbb{X}} f \, d\mu = \int_{\mathbb{X}} f^+ \, d\mu - \int_{\mathbb{X}} f^- \, d\mu.$$

*If both integrals are finite, we say that f is **integrable** (אינטגרבילית).*

Comment: Since $|f| = f^+ + f^-$, f is integrable if and only if

$$\int_{\mathbb{X}} |f| \, d\mu < \infty.$$

Proposition 3.32 The set of integrable functions forms a real vector space; the integral is a linear functional on that space.

Proof: Let f, g be integrable and let $a, b \in \mathbb{R}$. Since

$$|af + bg| \le |a||f| + |b||g|,$$

if follows from the monotonicity of the integral of non-negative functions that $af + bg$ is integrable. Next, suppose that $a > 0$. Then,

$$\int_{\mathbb{X}} af \, d\mu = \int_{\mathbb{X}} (af)^+ \, d\mu - \int_{\mathbb{X}} (af)^- \, d\mu = a \int_{\mathbb{X}} f^+ \, d\mu - a \int_{\mathbb{X}} f^- \, d\mu = a \int_{\mathbb{X}} f \, d\mu.$$

If $a < 0$ we proceed similarly, using the fact that $(af)^\pm = -af^\mp$. Finally, let $h = f + g$. Then,

$$h^+ - h^- = f^+ - f^- + g^+ - g^-,$$

which we re-organize as

$$h^+ + f^- + g^- = h^- + f^+ + g^+.$$

From the additivity of the integral for non-negative functions (Proposition 3.23),

$$\int_{\mathbb{X}} h^+ \, d\mu + \int_{\mathbb{X}} f^- \, d\mu + \int_{\mathbb{X}} g^- \, d\mu = \int_{\mathbb{X}} h^- \, d\mu + \int_{\mathbb{X}} f^+ \, d\mu + \int_{\mathbb{X}} g^+ \, d\mu,$$

hence

$$\begin{aligned}\int_{\mathbb{X}} h \, d\mu &= \int_{\mathbb{X}} h^+ \, d\mu - \int_{\mathbb{X}} h^- \, d\mu \\ &= \int_{\mathbb{X}} f^+ \, d\mu - \int_{\mathbb{X}} f^- \, d\mu + \int_{\mathbb{X}} g^+ \, d\mu - \int_{\mathbb{X}} g^- \, d\mu \\ &= \int_{\mathbb{X}} f \, d\mu + \int_{\mathbb{X}} g \, d\mu.\end{aligned}$$

■

Definition 3.33 Let $f : \mathbb{X} \to \mathbb{C}$ be measurable. It is called integrable if $|f|$ is integrable and we define

$$\int_{\mathbb{X}} f \, d\mu = \int_{\mathbb{X}} \operatorname{Re} f \, d\mu + \imath \int_{\mathbb{X}} \operatorname{Im} f \, d\mu.$$

The space of complex integrable functions is a complex vector space, which we denote by $L^1(\mu)$ (or, to avoid all ambiguity, $L^1(\mathbb{X}, \Sigma, \mu)$).

Proposition 3.34 If $f \in L^1(\mu)$ then

$$\left| \int_{\mathbb{X}} f \, d\mu \right| \le \int_{\mathbb{X}} |f| \, d\mu.$$

Proof: If f is real-valued, then

$$\left| \int_{\mathbb{X}} f \, d\mu \right| = \left| \int_{\mathbb{X}} f^+ \, d\mu - \int_{\mathbb{X}} f^- \, d\mu \right| \le \int_{\mathbb{X}} f^+ \, d\mu + \int_{\mathbb{X}} f^- \, d\mu = \int_{\mathbb{X}} |f| \, d\mu.$$

If f is complex-valued and its integral is zero then the statement is trivial; otherwise, there exists an $\alpha \in \mathbb{C}$ such that $\alpha \int_{\mathbb{X}} f \, d\mu$ is real, positive and $|\alpha| = 1$. Then,

$$\begin{aligned}\left|\int_{\mathbb{X}} f \, d\mu\right| &= \alpha \int_{\mathbb{X}} f \, d\mu = \operatorname{Re}\left(\alpha \int_{\mathbb{X}} f \, d\mu\right) \\ &= \operatorname{Re} \int_{\mathbb{X}} \alpha f \, d\mu = \int_{\mathbb{X}} \operatorname{Re}(\alpha f) \, d\mu \leq \int_{\mathbb{X}} |\alpha f| \, d\mu = \int_{\mathbb{X}} |f| \, d\mu,\end{aligned}$$

where the fourth passage follows from the definition of the integral of a complex function. ■

Proposition 3.35 If $f \in L^1(\mu)$ is real-valued then the set

$$\{x : f(x) \neq 0\}$$

is σ-finite. Moreover, for $f, g \in L^1(\mu)$

$$\int_A f \, d\mu = \int_A g \, d\mu \qquad \textit{for all } A \in \Sigma,$$

if and only if

$$\int_{\mathbb{X}} |f - g| \, d\mu = 0$$

if and only if $f = g$ μ-a.e.

Proof: The first assertion follows from Proposition 3.29, as

$$\{x : f(x) \neq 0\} = \{x : f^+(x) > 0\} \sqcup \{x : f^-(x) > 0\}.$$

Suppose that

$$\int_{\mathbb{X}} |f - g| \, d\mu = 0.$$

Then, by Proposition 3.25, $f = g$ μ-a.e., and for every $A \in \Sigma$,

$$\left|\int_A f \, d\mu - \int_A g \, d\mu\right| = \left|\int_{\mathbb{X}} \chi_A (f - g) \, d\mu\right| \leq \int_{\mathbb{X}} |f - g| \, d\mu = 0.$$

Conversely, suppose that the set $\{x : f(x) = g(x)\}$ is not a null set. Then, either

$$A = \{x : f(x) > g(x)\} \qquad \text{or} \qquad B = \{x : f(x) < g(x)\}$$

is not a null set. Suppose without loss of generality that A has positive measure; then,

$$A = \bigcup_{n=1}^{\infty} A_n \equiv \bigcup_{n=1}^{\infty} \{x \,:\, f(x) - g(x) > 1/n\},$$

and by the lower-semicontinuity of μ one of the A_n must have finite measure, $\mu(A_n) = c > 0$. Then,

$$\int_{A_n} f\,d\mu - \int_{A_n} g\,d\mu = \int_{A_n} (f-g)\,d\mu \geq \frac{c}{n} > 0.$$

—17h(2018)— ∎

The last proposition asserts that integrals are not affected by variations of the integrand on null sets. With this in mind, it is customary to redefine $L^1(\mu)$ as equivalence classes of integrable functions that are equal up to sets of measure zero. This approach has many advantages. For example, $L^1(\mu)$ is now a metric space, with

$$d(f,g) = \int_{\mathbb{X}} |f-g|\,d\mu.$$

Indeed, positivity is satisfied only if we identify functions that differ on null sets.

✎ *Exercise 3.11* Let $(\mathbb{X}, \Sigma, \mu)$ be a measure space. Let $A \in \Sigma$ satisfy $\mu(A) < \infty$ and let

$$\sup_{x \in A} f(x) = M < \infty.$$

Suppose that

$$\int_A f\,d\mu = M\,\mu(A).$$

Prove that $f(x) = M$ a.e. in A.

✎ *Exercise 3.12* Let (X, Σ, μ) be a measure space, and let $f \in L^1(\mu)$. Show that

$$\lim_{t\to\infty} t\,\mu(\{x \,:\, |f(x)| \geq t\}) = 0.$$

✎ *Exercise 3.13* Easier than the previous exercise: let (X, Σ, μ) be a measure space, and let $f \in L^1(\mu)$. Show that

$$\lim_{t\to\infty} \mu(\{x \,:\, |f(x)| \geq t\}) = 0.$$

✎ *Exercise 3.14* Let $(\mathbb{X}, \Sigma, \mu)$ be a measure space. Prove that μ is σ-finite if and only if there exists a strictly positive $f \in L^1(\mu)$.

Thus far, we have seen two convergence theorems: Monotone Convergence and Fatou's lemma. We now prove the third convergence theorem, which has many applications:

Theorem 3.36 (Dominated Convergence (התכנסות נשלטת)) Let $(\mathbb{X}, \Sigma, \mu)$ be a complete measure space. Let $f_n \in L^1(\mu)$ such that $f_n \to f$ μ-a.e). Moreover, suppose that there exists a non-negative function $g \in L^1(\mu)$ such that

$$|f_n| \le g \qquad \mu\text{-a.e. for all } n.$$

Then $f \in L^1(\mu)$ and

$$\lim_{n\to\infty} \int_{\mathbb{X}} |f_n - f|\, d\mu = 0.$$

In particular,

$$\lim_{n\to\infty} \int_{\mathbb{X}} f_n\, d\mu = \int_{\mathbb{X}} f\, d\mu.$$

Comment: The assumption in the Dominated Convergence Theorem is that the graphs of all f_n are confined to a region of finite measure.

Proof: f is measurable by Proposition 3.16 (Item 2). Since $|f| \le g$ μ-a.e. and $g \in L^1(\mu)$ it follows that $f \in L^1(\mu)$. It suffices to consider the case where f is real-valued; since

$$2g - |f - f_n| \ge 0,$$

it follows from Fatou's lemma that

$$\begin{aligned}
\int_{\mathbb{X}} 2g\, d\mu &\le \liminf_{n\to\infty} \int_{\mathbb{X}} (2g - |f - f_n|)\, d\mu \\
&= \int_{\mathbb{X}} 2g\, d\mu + \liminf_{n\to\infty} \left(- \int_{\mathbb{X}} |f - f_n|\, d\mu \right) \\
&= \int_{\mathbb{X}} 2g\, d\mu - \limsup_{n\to\infty} \int_{\mathbb{X}} |f - f_n|\, d\mu,
\end{aligned}$$

and using the fact that g is integrable,

$$\limsup_{n\to\infty} \int_{\mathbb{X}} |f - f_n|\, d\mu \le 0,$$

which completes the proof. ∎

TA material 3.2 The need for a dominating function: Take the following example: the measure space is $(\mathbb{R}, \mathscr{B}(\mathbb{R}), m)$, and

$$f_n = \frac{1}{2n} \chi_{(n^2-n, n^2+n)}.$$

This function converges uniformly (and hence everywhere) to $f = 0$, and yet

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, dm = 1 \neq 0 = \int_{\mathbb{R}} f \, dm.$$

Note that there is no *integrable* g that dominates the f_n.

✎ *Exercise 3.15* Let $f_n, f, g_n, g : \mathbb{X} \to \mathbb{R}$ be measurable functions in a measure space $(\mathbb{X}, \Sigma, \mu)$. Suppose that $g_n, g \in L^1(\mu)$, $|f_n| \leq g_n$,

$$\lim_{n\to\infty} f_n = f \qquad \text{and} \qquad \lim_{n\to\infty} g_n = g \quad \text{a.e.,}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{X}} g_n \, d\mu = \int_{\mathbb{X}} g \, d\mu.$$

Prove that $f \in L^1(\mu)$ and

$$\lim_{n\to\infty} \int_{\mathbb{X}} f_n \, d\mu = \int_{\mathbb{X}} f \, d\mu.$$

✎ *Exercise 3.16* Let $(\mathbb{X}, \Sigma, \mu)$ be a measure space, and let $f_n \in L^1(\mu)$ be a sequence of non-negative functions, converging pointwise to $f \in L^1(\mu)$. Prove that

$$\lim_{n\to\infty} \left(\int_{\mathbb{X}} f_n \, d\mu - \int_{\mathbb{X}} f \, d\mu - \int_{\mathbb{X}} |f - f_n| \, d\mu \right) = 0.$$

—18h(2017)—

Proposition 3.37 Let $f_n \in L^1(\mu)$ such that

$$\sum_{n=1}^{\infty} \int_{\mathbb{X}} |f_n| \, d\mu < \infty.$$

Then, $\sum_{n=1}^{\infty} f_n$ converges μ-a.e. to a function in $L^1(\mu)$ and

$$\int_{\mathbb{X}} \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int_{\mathbb{X}} f_n \, d\mu.$$

Proof: By Proposition 3.24, since $|f_n| \in L^+(\mathbb{X}, \Sigma)$,

$$\int_{\mathbb{X}} \sum_{n=1}^{\infty} |f_n| \, d\mu = \sum_{n=1}^{\infty} \int_{\mathbb{X}} |f_n| \, d\mu.$$

(Note that in Proposition 3.24 both sides could be infinite, however here they are finite.) Thus,

$$g \equiv \sum_{n=1}^{\infty} |f_n| \in L^1(\mu),$$

and by Proposition 3.35 it is finite μ-a.e. For each such x, the sequence of functions

$$F_n(x) = \sum_{k=1}^{n} f_k(x)$$

converges and satisfies $|F_n| \le g$. Applying the Dominated Convergence Theorem to the sequence of partial sums, we obtain

$$\lim_{n\to\infty} \int_{\mathbb{X}} F_n \, d\mu = \int_{\mathbb{X}} \lim_{n\to\infty} F_n \, d\mu,$$

which is the desired result. ■

Proposition 3.38 (Simple functions are dense in $L^1(\mu)$) *Let $f \in L^1(\mu)$. For every $\varepsilon > 0$ there exists a simple function*

$$\phi = \sum_{k=1}^{n} a_k \chi_{A_k},$$

such that

$$\int_{\mathbb{X}} |f - \phi| \, d\mu < \varepsilon.$$

Proof: We may construct ϕ_n as in Theorem 3.15, such that it converges from below to f^+ and from above to $(-f^-)$. Then, $|f - \phi_n| \to 0$ everywhere f is finite, i.e., a.e. Since

$$|f - \phi_n| \le |f| + |\phi_n| \le 2|f|,$$

it follows from Dominated Convergence that

$$\lim_{n\to\infty} \int_{\mathbb{X}} |f - \phi_n| \, d\mu = \int_{\mathbb{X}} \lim_{n\to\infty} |f - \phi_n| \, d\mu = 0.$$

■

3.5 Lebesgue and Riemann integration

In the particular case where the measure space is $(\mathbb{R}, \mathcal{L}, m)$, the integral we defined is called the **Lebesgue integral**. The integral defined in first-year calculus is the **Riemann integral**. We will now examine the relation between the two.

Let $\mathbb{X} = [a, b]$ and let $f : [a, b] \to \mathbb{R}$ be bounded. If $P = \{t_0, \dots, t_n\}$ is a partition of that interval, then we define

$$S_P(f) = \sum_{k=1}^{n} M_k(t_k - t_{k-1}) \qquad \text{and} \qquad s_P(f) = \sum_{k=1}^{n} m_k(t_k - t_{k-1}),$$

where

$$M_k = \sup_{t_{k-1} \le x \le t_k} f(x) \qquad \text{and} \qquad m_k = \inf_{t_{k-1} \le x \le t_k} f(x),$$

and

$$\overline{I}(f) = \inf_P S_P(f) \qquad \text{and} \qquad \underline{I}(f) = \sup_P s_P(f)$$

f is Riemann-integrable if both are equal.

Theorem 3.39 (Riemann integrability implies Lebesgue integrability) If f is Riemann-integrable on $[a, b]$ then it is Lebesgue-measurable and therefore Lebesgue-integrable, with

$$\int_a^b f(x)\, dx = \int_{[a,b]} f\, dm.$$

Moreover, f is Riemann-integrable if and only if the set of points in which it is discontinuous has measure zero.

Proof: Suppose that f is Riemann-integrable. For every partition P, define

$$G_P = \sum_{k=1}^{m} M_k \chi_{[t_{k-1}, t_k]} \qquad \text{and} \qquad g_P = \sum_{k=1}^{m} m_k \chi_{[t_{k-1}, t_k]}.$$

We note that

$$S_P(f) = \int_{[a,b]} G_P\, dm \qquad \text{and} \qquad s_P(f) = \int_{[a,b]} g_P\, dm.$$

Also, for every partition,

$$g_P \le f \le G_P.$$

By the construction of the Riemann integral, there is a sequence of partitions P_n, such that G_{P_n} is monotonically decreasing, g_{P_n} is monotonically increasing, and

$$\lim_{n\to\infty} \int_{[a,b]} G_{P_n}\, dm = \lim_{n\to\infty} \int_{[a,b]} g_{P_n}\, dm = \int_a^b f(x)\, dx.$$

Let

$$G = \lim_{n\to\infty} G_{P_n} \qquad \text{and} \qquad g = \lim_{n\to\infty} g_{P_n}.$$

Then, by Dominated Convergence,

$$\int_{[a,b]} G\, dm = \lim_{n\to\infty} \int_{[a,b]} G_{P_n}\, dm = \int_a^b f(x)\, dx$$
$$\int_{[a,b]} g\, dm = \lim_{n\to\infty} \int_{[a,b]} g_{P_n}\, dm = \int_a^b f(x)\, dx.$$

i.e.,

$$\int_{[a,b]} (G - g)\, dm = 0.$$

Since

$$g \le f \le G,$$

it follows that $G = g$ m-a.e., hence f is measurable (here we use the fact that m is complete), and

$$\int_{[a,b]} f\, dm = \int_{[a,b]} G\, dm = \int_a^b f(x)\, dx.$$

■

✎ *Exercise 3.17* Prove the second part of this theorem: that a function is Riemann-integrable of an only if the set of points at which it is discontinuous has measure zero.

3.6 Modes of convergence

Let $\mathbb{X}$ be a set (with no additional structure) and let $f_n, f : \mathbb{X} \to \mathbb{C}$. There are two classical senses in which we may define convergence $f_n \to f$:

(a) **Pointwise convergence**:

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad \forall x \in \mathbb{X}.$$

(b) **Uniform convergence**:

$$\lim_{n\to\infty} \sup_{x\in\mathbb{X}} |f_n(x) - f(x)| = 0.$$

Uniform convergence implies pointwise convergence, but the opposite is not true: The classical example is $\mathbb{X} = [0,1)$ and $f_n(x) = x^n$, which converges to $f(x) = 0$ pointwise, but not uniformly.

If $(\mathbb{X}, \Sigma, \mu)$ is a measure space, then we may define additional modes of convergence (for measurable functions):

(d) μ**-almost-everywhere convergence**:

$$\mu\left(\{x : \lim_{n\to\infty} f_n(x) \neq f(x)\}\right) = 0.$$

(e) $L^1(\mu)$**-convergence**:

$$\lim_{n\to\infty} \int_{\mathbb{X}} |f_n - f| \, d\mu = 0.$$

(f) **Convergence in measure**: for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

—19h(2017)—

Examples: Consider the following functions:

$$f_n = \frac{\chi_{(0,n)}}{n} \qquad g_n = \chi_{(n,n+1)} \qquad h_n = n\chi_{[0,1/n]}$$

$$p_1 = \chi_{[0,1]} \quad p_2 = \chi_{[0,1/2]} \quad p_3 = \chi_{[1/2,1]} \quad p_4 = \chi_{[0,1/4]}, \dots.$$

	uniformly	pointwise	μ-a.e.	$L^1(\mu)$	in measure
$f_n \to 0$	yes	yes	yes	no	yes
$g_n \to 0$	no	yes	yes	no	no
$h_n \to 0$	no	no	yes	no	yes
$p_n \to 0$	no	no	no	yes	yes

Proposition 3.40 Convergence in L^1 implies convergence in measure.

Proof: Suppose that $f_n \to f$ in $L^1(\mu)$, i.e.,

$$\lim_{n\to\infty} \int_{\mathbb{X}} |f_n - f| \, d\mu = 0.$$

Let $\varepsilon > 0$ be given, and denote

$$A_n = \{x \, : \, |f_n(x) - f(x)| \geq \varepsilon\}.$$

Then,

$$\int_{\mathbb{X}} |f_n - f| \, d\mu \geq \int_{A_n} |f_n - f| \, d\mu \geq \varepsilon \mu(A_n),$$

i.e.,

$$\mu(\{x \, : \, |f_n(x) - f(x)| \geq \varepsilon\}) \leq \frac{1}{\varepsilon} \int_{\mathbb{X}} |f_n - f| \, d\mu \to 0.$$

■

Proposition 3.41 If $f_n \to f$ in measure, then f_n has a subsequence converging to f μ-a.e.

Proof: By the definition of convergence in measure, for every $\varepsilon > 0$

$$\lim_{n\to\infty} \mu(\{x \, : \, |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

Take $\varepsilon = 1/k$. Then, there exists an n_k such that

$$\mu\left(\{x \, : \, |f_{n_k}(x) - f(x)| \geq 1/k\}\right) \leq \frac{1}{k^2}.$$

Since the right-hand side is summable, it follows from Borel-Cantelli (Proposition 3.30) that

$$\mu(\{x \, : \, |f_{n_k}(x) - f(x)| \geq 1/k \text{ for infinitely many } k\text{'s}\}) = 0,$$

whereas

$$\begin{aligned}
&\{x \, : \, |f_{n_k}(x) - f(x)| \geq 1/k \text{ for infinitely many } k\text{'s}\}^c \\
&\quad = \{x \, : \, |f_{n_k}(x) - f(x)| < 1/k \text{ for } k \text{ large enough}\} \\
&\quad \subset \{x \, : \, \lim_{k\to\infty} f_{n_k}(x) = f(x)\}.
\end{aligned}$$

That is, $f_{n_k} \to f$ μ-a.e. ■

Corollary 3.42 If $f_n \to f$ in L^1, then there is a subsequence of f_n converging to f μ-a.e.

Proof: L^1-convergence implies convergence in measure, which implies the a.e. convergence of a subsequence. ■

—19h(2018)—

✎ *Exercise 3.18* Let $(\mathbb{X}, \Sigma, \mu)$ be a σ-finite measure space. Let $f_n \in L^1(\mu)$ be non-negative functions satisfying

$$\int_{\mathbb{X}} f_n \, d\mu = 1.$$

(a) Show that it is not necessarily true that $f_n/n \to 0$ a.e. (b) Show that $f_n/n^2 \to 0$ a.e.

✎ *Exercise 3.19* Let $(\mathbb{X}, \Sigma, \mu)$ be a finite measure space and let f_n, f be uniformly-bounded measurable real-valued functions. Show that $f_n \to f$ in measure implies that $f_n \to f$ in $L^1(\mu)$.

✎ *Exercise 3.20* Show that the condition

$$\lim_{n \to \infty} \mu(\{x : |f_n(x) - f(x)| > 0\}) = 0,$$

implies that $f_n \to f$ in measure.

✎ *Exercise 3.21* Show that if $f_n \le f_{n+1}$ is a monotone sequence of measurable functions and $f_n \to f$ in measure, then $f_n \to f$ a.e.

✎ *Exercise 3.22* Show that Dominated Convergence holds if a.e. convergence is replaced by convergence in measure: let $(\mathbb{X}, \Sigma, \mu)$ be a measure space. Let f_n, f be measurable, with $|f_n| \le g \in L^1(\mu)$. Furthermore, $f_n \to f$ in measure. Show that $f \in L^1(\mu)$ and

$$\lim_{n \to \infty} \int_{\mathbb{X}} f_n \, d\mu = \int_{\mathbb{X}} f \, d\mu.$$

✎ *Exercise 3.23* Show that $f_n \to f$ and $g_n \to g$ a.e., then $f_n + g_n \to f + g$ a.e.

The final theorem of this section applies to finite measure spaces, and states that in such spaces convergence a.e. implies, in a certain sense, "almost uniform convergence".

Theorem 3.43 (Egoroff) *Let $(\mathbb{X}, \Sigma, \mu)$ be a finite measure space. Let $f_n, f : \mathbb{X} \to \mathbb{C}$ be measurable, such that*

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad \mu\text{-a.e.}$$

Then there exists for every $\varepsilon > 0$ a measurable set A, such that $\mu(A) < \varepsilon$, and $f_n \to f$ uniformly on A^c.

Proof: Let

$$B = \{x \,:\, \lim_{n\to\infty} f_n(x) = f(x)\},$$

which is a set of full-measure (i.e., its complement is a null-set).

For every n and k we define the set

$$A(n,k) = \bigcup_{j=n}^{\infty} \{x \in B \,:\, |f_j(x) - f(x)| \ge 1/k\}.$$

This collection is decreasing as a function on n and increasing as a function of k. Since $f_n \to f$ pointwise in B,

$$\bigcap_{n=1}^{\infty} A(n,k) = \varnothing$$

(this is the set of points x for which $|f_n(x) - f(x)| \ge 1/k$ infinitely-often). Since μ is a finite measure, it follows from the upper-semicontinuity of the measure that for every $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \mu(A(n,k)) = \mu\left(\bigcap_{n=1}^{\infty} A(n,k)\right) = 0.$$

Given $\varepsilon > 0$ and $k \in \mathbb{N}$, we may choose n_k sufficiently large such that

$$\mu(A(n_k,k)) \le \varepsilon 2^{-k}.$$

Let then

$$A = B^c \cup \bigcup_{k=1}^{\infty} A(n_k,k).$$

By the sub-additivity of μ, $\mu(A) < \varepsilon$. On the other hand,

$$\begin{aligned} A^c &= B \cap \left(\bigcup_{k=1}^{\infty} A(n_k, k) \right)^c \\ &= B \cap \bigcap_{k=1}^{\infty} A^c(n_k, k) \\ &= B \cap \bigcap_{k=1}^{\infty} \bigcap_{j=n_k}^{\infty} \{x \in B : |f_j(x) - f(x)| \geq 1/k\}^c \\ &= \bigcap_{k=1}^{\infty} \bigcap_{j=n_k}^{\infty} \{x \in B : |f_j(x) - f(x)| < 1/k\}. \end{aligned}$$

That is, $x \in A^c$ implies that

$$\forall k \in \mathbb{N} \quad \exists n_k \quad \forall j > n_k \qquad |f_j(x) - f(x)| < 1/k.$$

In other words, $f_n \to f$ uniformly on A^c. ■

✎ *Exercise 3.24* We say that $f_n \to f$ **almost-uniformly** if there exists for every $\varepsilon > 0$ a measurable set A, such that $\mu(A) < \varepsilon$ and f_n converges to f uniformly on A^c. Show that almost uniform convergence implies a.e. convergence (hence convergence in measure).

✎ *Exercise 3.25* Show that Egorov's theorem does not extend to σ-finite spaces.

The following theorem states that in a certain sense, measurable functions over finite segment are "almost continuous":

Theorem 3.44 (Lusin) Let $f : [a,b] \to \mathbb{R}$ be measurable. Then, for every $\varepsilon > 0$ there exists a compact set $K \subset [a,b]$ such that $m([a,b] \smallsetminus K]) < \varepsilon$, and a function $g \in C([a,b])$, such that $g|_K = f|_K$.

Proof: It suffices to consider the case where f is non-negative (for then, apply the theorem for $f^{\pm}$). Then, there exists by Theorem 3.15 a sequence of simple function $\phi_n \in \mathrm{SF}^+(\mathbb{X}, \Sigma)$, such that

$$\lim_{n\to\infty} \psi_n(x) = f(x) \qquad \forall x \in [a,b].$$

Now, every simple function is of the form

$$\psi_n = \sum_{i=1}^{m_n} \alpha_{n,i} \chi_{A_{n,i}}.$$

Let $\varepsilon > 0$ be given and let $\phi_{n,i} \in C([a,b])$ satisfy

$$m(\{x : \phi_{n,i}(x) \neq \chi_{A_{n,i}}(x)\}) < \frac{\varepsilon}{4m_n 2^n}.$$

Then,

$$f_n = \sum_{i=1}^{m_n} \alpha_{n,i}\phi_{n,i}$$

is continuous and satisfies,

$$m(\{x : f_n(x) \neq \psi_n(x)\}) < \frac{\varepsilon}{4\,2^n}.$$

Finally, set

$$E = \bigcup_{n=1}^{\infty} \{x : f_n(x) \neq \psi_n(x)\}.$$

Then, $m(E) < \varepsilon/4$ and

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad \forall x \in [a,b] \setminus E.$$

By Egorov's theorem, there exists a measurable set $A \subset [a,b] \setminus E$ such that f_n converges uniformly of A and $m([a,b] \setminus E \setminus A) < \varepsilon/4$. Its limit is continuous (relative to the subspace topology of $[a,b] \setminus E \setminus A$).

Let $K \subset [a,b] \setminus E \setminus A$ be compact, such that

$$m([a,b] \setminus K) < \varepsilon$$

(here we use the inner-regularity of the Lebesgue measure). Note that $[a,b] \setminus K$ is open, hence is a countable union of open intervals. We can therefore extend $f|_K$ into a continuous function g by linear interpolation in each interval in $[a,b] \setminus K$. Clearly,

$$m(\{x : g(x) \neq f(x)\}) < e.$$

■

3.7 Product measures

Let $(\mathbb{X}, \Sigma_{\mathbb{X}}, \mu)$ and $(\mathbb{Y}, \Sigma_{\mathbb{Y}}, \nu)$ be measure spaces. We have already defined the product σ-algebra,

$$\Sigma_{\mathbb{X}} \otimes \Sigma_{\mathbb{Y}}$$

on $\mathbb{X} \times \mathbb{Y}$. We proceed to define a product measure on the product σ-algebra. The construction is quite natural. We want a measure, which we will denote by $\mu \times \nu$, with respect to which sets of the form $A \times B$ are "independent" (in the probabilistic sense), i.e.,

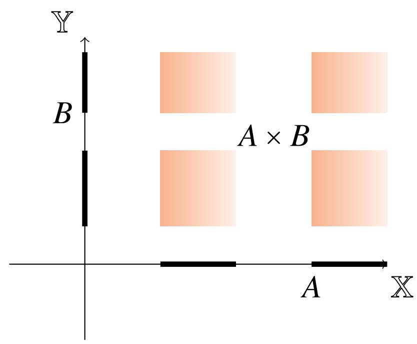
$$\mu \times \nu(A \times B) = \mu(A)\nu(B).$$

With this in mind, we start constructing such measures.

Definition 3.45 Let $(\mathbb{X}, \Sigma_{\mathbb{X}})$ *and* $(\mathbb{Y}, \Sigma_{\mathbb{Y}})$ *be measurable spaces. A* ***measurable rectangle*** *(מלבן מדיד) in* $\mathbb{X} \times \mathbb{Y}$ *is a set of the form*

$$A \times B \qquad A \in \Sigma_{\mathbb{X}}, \quad B \in \Sigma_{\mathbb{Y}}.$$

(Note that even for $\mathbb{X} = \mathbb{Y} = \mathbb{R}$*, it needs not look like a rectangle.)*



Lemma 3.46 The collection $\mathcal{E}$ *of measurable rectangles is an elementary family.*

Proof: We need to show that $\mathcal{E}$ is closed under intersection and that the complement of a measurable rectangle is a finite disjoint union of elements in $\mathcal{E}$. Indeed,

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

and

$$(A \times B)^c = (A \times B^c) \amalg (A^c \times B) \amalg (A^c \times B^c).$$

■

Corollary 3.47 The collection $\mathcal{A}$ of finite disjoint unions of elementary rectangles forms an algebra.

Proposition 3.48 Let $\mathcal{A}$ be the algebra of finite disjoint unions of elementary rectangles. Then,

$$\sigma(\mathcal{A}) = \Sigma_{\mathbb{X}} \otimes \Sigma_{\mathbb{Y}}.$$

Proof: Since for every $A \in \Sigma_{\mathbb{X}}$ and $B \in \Sigma_{\mathbb{Y}}$,

$$A \times \mathbb{Y} \in \mathcal{E} \subset \mathcal{A} \qquad \text{and} \qquad \mathbb{X} \times B \in \mathcal{E} \subset \mathcal{A}$$

if follows that

$$\Sigma_{\mathbb{X}} \otimes \Sigma_{\mathbb{Y}} = \sigma(\{A \times \mathbb{Y} : \ : A \in \Sigma_{\mathbb{X}}\} \cup \{\mathbb{X} \times B \ : \ B \in \Sigma_{\mathbb{Y}}\}) \subset \sigma(\mathcal{A}).$$

Conversely, $\mathcal{A} \subset \Sigma_{\mathbb{X}} \otimes \Sigma_{\mathbb{Y}}$, hence $\sigma(\mathcal{A}) \subset \Sigma_{\mathbb{X}} \otimes \Sigma_{\mathbb{Y}}$. ■

We next want to define a pre-measure π on $\mathcal{A}$. Naturally, we will set

$$\pi\left(\coprod_{k=1}^{n} A_k \times B_k\right) = \sum_{k=1}^{n} \mu(A_k)\nu(B_k).$$

As in previous instances, we need to show that this definition is independent of representation, and that π is σ-additive. —20h(2018)—

Independence on representation is straightforward. To prove that π is σ-additive, it suffices to consider the case where $A_n \times B_n$ are disjoint measurable rectangles, with

$$\coprod_{n=1}^{\infty} A_n \times B_n = A \times B.$$

For $x \in \mathbb{X}$ and $y \in \mathbb{Y}$,

$$\chi_A(x)\chi_B(y) = \sum_{n=1}^{\infty} \chi_{A_n}(x)\chi_{B_n}(y).$$

Integrating over x (viewing y as fixed), using Proposition 3.24 (for series of functions in $L^+(\mathbb{X}, \Sigma)$),

$$\mu(A)\chi_B(y) = \int_{\mathbb{X}} \sum_{n=1}^{\infty} \chi_{A_n}\chi_{B_n}(y)\, d\mu = \sum_{n=1}^{\infty} \int_{\mathbb{X}} \chi_{B_n}(y)\chi_{A_n}\, d\mu = \sum_{n=1}^{\infty} \chi_{B_n}(y)\mu(A_n).$$

Similarly, integrating over y we obtain that

$$\pi(A \times B) = \mu(A)\nu(B) = \sum_{n=1}^{\infty} \mu(A_n)\nu(B_n) = \sum_{n=1}^{\infty} \pi(A_n \times B_n).$$

The pre-measure π, which is defined on the algebra $\mathcal{A}$ generates an outer measure, whose restriction on $\Sigma_{\mathbb{X}} \otimes \Sigma_{\mathbb{Y}}$ is a measure extending π. We call this measure the **product measure** (מידת מכפלה), which we denote by $\mu \times \nu$. Note that if both μ and ν are σ-finite, i.e.,

$$\mathbb{X} = \bigcup_{n=1}^{\infty} A_n \quad \text{and} \quad \mathbb{Y} = \bigcup_{n=1}^{\infty} B_n, \qquad \mu(A_n), \nu(B_n) < \infty,$$

then

$$\mathbb{X} \times \mathbb{Y} = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_n \times B_m,$$

and

$$\pi(A_n \times B_m) = \mu(A_m)\nu(B_n) < \infty,$$

i.e., π is σ-finite, hence it has unique extension.

Comment: The construction of a product can be extended to any finite number of
—21h(2017)— factors.

Definition 3.49 *Let $\mathbb{X}$ and $\mathbb{Y}$ be sets and let $A \subset \mathbb{X} \times \mathbb{Y}$. For $x \in \mathbb{X}$, the **x-section** (חתך) of A is*

$$A_x = \{y \in \mathbb{Y} \,:\, (x, y) \in A\} \subset \mathbb{Y}.$$

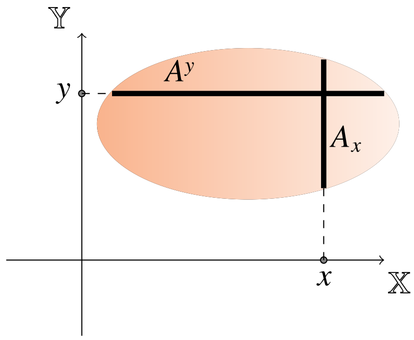
Likewise, for $y \in \mathbb{Y}$, the y-section of A is

$$A^y = \{x \in \mathbb{X} \,:\, (x, y) \in A\} \subset \mathbb{X}.$$

For a function on $\mathbb{X} \times \mathbb{Y}$ we define its x-section $f_x : \mathbb{Y} \to \mathbb{R}$ and its y-section $f^y : \mathbb{X} \to \mathbb{R}$ by

$$f_x(y) = f(x, y) \qquad \text{and} \qquad f^y(x) = f(x, y).$$

The x-section of f is the formal way of saying "we fix x and consider f only as a function of y".



Lemma 3.50 *Let $A \subset \mathbb{X} \times \mathbb{Y}$. Then, for $x \in \mathbb{X}$,*

$$(\chi_A)_x = \chi_{A_x} : \mathbb{Y} \to \mathbb{R}.$$

Likewise, for $y \in \mathbb{Y}$,

$$(\chi_A)^y = \chi_{A^y} : \mathbb{X} \to \mathbb{R}.$$

Proof: Let $x \in \mathbb{X}$. Then,

$$(\chi_A)_x(y) = \chi_A(x, y) = \begin{cases} 1 & (x, y) \in A \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & y \in A_x \\ 0 & \text{otherwise} \end{cases} = \chi_{A_x}(y).$$

■

The following proposition shows that sections of measurable sets/functions and measurable.

Proposition 3.51 *Let $(\mathbb{X}, \Sigma_{\mathbb{X}})$ and $(\mathbb{Y}, \Sigma_{\mathbb{Y}})$ be measurable spaces.*

1. *If $E \in \Sigma_{\mathbb{X}} \otimes \Sigma_{\mathbb{Y}}$ then $E_x \in \Sigma_{\mathbb{Y}}$ and $E^y \in \Sigma_{\mathbb{X}}$ for all $x \in \mathbb{X}$ and $y \in \mathbb{Y}$.*
2. *If $f : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}$ is $\Sigma_{\mathbb{X}} \otimes \Sigma_{\mathbb{Y}}$-measurable, then f_x is $\Sigma_{\mathbb{Y}}$-measurable and f^y is $\Sigma_{\mathbb{X}}$-measurable for all $x \in \mathbb{X}$ and $y \in \mathbb{Y}$.*

Proof: Given $x \in \mathbb{X}$ and $y \in \mathbb{Y}$, define

$$\mathcal{R} = \{E \subset \mathbb{X} \times \mathbb{Y} \ : \ E_x \in \Sigma_{\mathbb{Y}} \quad \text{and} \quad E^y \in \Sigma_{\mathbb{X}}\}.$$

$\mathcal{R}$ contains all the measurable rectangles: for $A \in \Sigma_{\mathbb{X}}$ and $B \in \Sigma_{\mathbb{Y}}$,

$$(A \times B)_x = \begin{cases} B & x \in A \\ \varnothing & \text{otherwise} \end{cases},$$

i.e., in either case, $(A \times B)_x \in \Sigma_{\mathbb{Y}}$.

It is easy to see that $\mathcal{R}$ is a σ-algebra: for example, if $E \in \mathcal{R}$, then

$$\begin{aligned} (E^c)_x &= \{y \in \mathbb{Y} \ : \ (x,y) \notin E\} = (E_x)^c \in \Sigma_{\mathbb{Y}} \\ (E^c)^y &= \{x \in \mathbb{X} \ : \ (x,y) \notin E\} = (E^y)^c \in \Sigma_{\mathbb{X}}, \end{aligned}$$

i.e., $E^c \in \mathcal{R}$.

Since $\mathcal{R}$ contains a collection generating $\Sigma_{\mathbb{X}} \otimes \Sigma_{\mathbb{Y}}$,

$$\Sigma_{\mathbb{X}} \otimes \Sigma_{\mathbb{Y}} \subset \mathcal{R},$$

which proves the first part.

The second part follows from the fact that if f is $\Sigma_{\mathbb{X}} \otimes \Sigma_{\mathbb{Y}}$-measurable, then for every $C \in \mathscr{B}(\mathbb{R})$,

$$f^{-1}(C) \in \Sigma_{\mathbb{X}} \otimes \Sigma_{\mathbb{Y}},$$

hence

$$\begin{aligned} (f_x)^{-1}(C) &= \{y \in \mathbb{Y} \ : \ f_x(y) \in C\} \\ &= \{y \in \mathbb{Y} \ : \ f(x,y) \in C\} \\ &= \{y \in \mathbb{Y} \ : \ (x,y) \in f^{-1}(C)\} \\ &= \{y \in \mathbb{Y} \ : \ y \in (f^{-1}(C))_x\} \\ &= (f^{-1}(C))_x \in \Sigma_{\mathbb{Y}}, \end{aligned}$$

—22h(2017)— where the last inclusion follows the first part. ■

Theorem 3.52 Let $(\mathbb{X}, \Sigma_{\mathbb{X}}, \mu)$ *and* $(\mathbb{Y}, \Sigma_{\mathbb{Y}}, \nu)$ *be* σ*-finite measure spaces. For* $E \in \Sigma_{\mathbb{X}} \otimes \Sigma_{\mathbb{Y}}$*, the real-valued functions* $f_E : \mathbb{X} \to \mathbb{R}$ *and* $g_E : \mathbb{Y} \to \mathbb{R}$ *defined by*

$$f_E : x \mapsto \nu(E_x) \qquad \textit{and} \qquad g_E : y \mapsto \mu(E^y)$$

are $\Sigma_{\mathbb{X}}$*- and* $\Sigma_{\mathbb{Y}}$*-measurable, respectively. Moreover,*

$$\mu \times \nu(E) = \int_{\mathbb{X}} f_E \, d\mu = \int_{\mathbb{Y}} g_E \, d\nu,$$

which we may also write as

$$\mu \times \nu(E) = \int_{\mathbb{X}} \nu(E_x) \, d\mu(x) = \int_{\mathbb{Y}} \mu(E^y) \, d\nu(y).$$

Proof: Start by assuming that both spaces are finite measure spaces. Let $\mathcal{C}$ be the class of sets for which the statements are true. Note that for each measurable rectangle, $E = A \times B$,

$$f_E(x) = \nu(E_x) = \begin{cases} \nu(B) & x \in A \\ 0 & \text{otherwise} \end{cases} = \nu(B)\chi_A(x)$$

and

$$g_E(y) = \mu(E^y) = \begin{cases} \mu(A) & y \in B \\ 0 & \text{otherwise} \end{cases} = \mu(A)\chi_B(y)$$

are both measurable. By definition of the product measure,

$$\int_{\mathbb{X}} f_E \, d\mu = \mu(A)\nu(B) \qquad \text{and} \qquad \int_{\mathbb{Y}} g_E \, d\nu = \mu(A)\nu(B).$$

Thus, the class $\mathcal{C}$ contains the measurable rectangles, which are a generating set for $\Sigma_{\mathbb{X}} \otimes \Sigma_{\mathbb{Y}}$. —21h(2018)—

By additivity, any finite disjoint union of measurable rectangles is also in $\mathcal{C}$. Indeed, let $E, F \in \Sigma_{\mathbb{X}} \otimes \Sigma_{\mathbb{Y}}$ be disjoint measurable rectangles. Then,

$$\begin{aligned} f_{E \amalg F}(x) &= \nu((E \amalg F)_x) = \nu(E_x) + \nu(F_x) = f_E(x) + f_F(x) \\ g_{E \amalg F}(y) &= \mu((E \amalg F)^y) = \mu(E^y) + \nu(F^y) = g_E(y) + g_F(y), \end{aligned}$$

proving that $f_{E\sqcup F}$ and $g_{E\sqcup F}$ are both measurable. Moreover,

$$\begin{aligned}\mu\times\nu(E\sqcup F) &= \mu\times\nu(E)+\mu\times\nu(F)\\ &= \int_{\mathbb{X}} f_E\,d\mu + \int_{\mathbb{X}} f_F\,d\mu = \int_{\mathbb{X}} f_{E\sqcup F}\,d\mu\\ &= \int_{\mathbb{Y}} g_E\,d\nu + \int_{\mathbb{Y}} g_F\,d\nu = \int_{\mathbb{Y}} g_{E\sqcup F}\,d\nu.\end{aligned}$$

This proves that $\mathcal{C}$ is an algebra. By the Monotone Class Theorem (Theorem 2.20), it suffices to prove that $\mathcal{C}$ is a monotone class (which will imply that it is a σ-algebra, hence equal to $\Sigma_{\mathbb{X}}\otimes\Sigma_{\mathbb{Y}}$).

So let (E_n) be an increasing sequence in $\mathcal{C}$ and let $E=\bigcup_{n=1}^\infty E_n$. The functions

$$f_{E_n}(x)=\nu((E_n)_x) \qquad \text{and} \qquad g_{E_n}(y)=\mu((E_n)^y)$$

are measurable (since $E_n\in\mathcal{C}$), increase pointwise, and by the lower-semicontinuity of measures converge (pointwise) to

$$\begin{aligned}\lim_{n\to\infty} f_{E_n}(x) &= \lim_{n\to\infty}\nu((E_n)_x)=\nu\left(\bigcup_{n=1}^\infty (E_n)_x\right)=\nu(E_x)=f_E(x)\\ \lim_{n\to\infty} g_{E_n}(x) &= \lim_{n\to\infty}\mu((E_n)^y)=\mu\left(\bigcup_{n=1}^\infty (E_n)^y\right)=\nu(E^y)=g_E(y).\end{aligned}$$

Then, f_E and g_E are measurable, and by Monotone Convergence,

$$\int_{\mathbb{X}} f_E(x)\,d\mu(x)=\lim_{n\to\infty}\int_{\mathbb{X}} f_{E_n}(x)\,d\mu(x)=\lim_{n\to\infty}\mu\times\nu(E_n)=\mu\times\nu(E),$$

where we used once again the lower-semicontinuity of measures. Similarly,

$$\int_{\mathbb{Y}} g_E(y)\,d\nu(y)=\lim_{n\to\infty}\int_{\mathbb{Y}} g_{E_n}(y)\,d\mu(y)=\lim_{n\to\infty}\mu\times\nu(E_n)=\mu\times\nu(E).$$

Hence $E\in\mathcal{C}$. A similar analysis holds for decreasing sequences, proving that $\mathcal{C}$ is a monotone class (which is where we use the fact that the measure is finite).

It remains to address the case where the spaces are σ-finite (see exercise). ■

✎ *Exercise 3.26* Extend the above theorem for the case of σ-finite spaces.

Theorem 3.53 (Fubini-Tonelli) *Let* $(\mathbb{X}, \Sigma_{\mathbb{X}}, \mu)$ *and* $(\mathbb{Y}, \Sigma_{\mathbb{Y}}, \nu)$ *be* σ*-finite measure spaces. Then,*

1. *Tonelli: Let* $f : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}$ *be in* $L^+(\mathbb{X} \times \mathbb{Y})$*. Then,*

$$g(x) = \int_{\mathbb{Y}} f_x(y)\, d\nu(y) \qquad \text{and} \qquad h(y) = \int_{\mathbb{X}} f^y(x)\, d\mu(x)$$

are in $L^+(\mathbb{X}, \Sigma_{\mathbb{X}})$ *and* $L^+(\mathbb{Y}, \Sigma_{\mathbb{Y}})$*, respectively. Moreover,*

$$\int_{\mathbb{X}\times\mathbb{Y}} f\, d(\mu\times\nu) = \int_{\mathbb{X}} \left(\int_{\mathbb{Y}} f_x(y)\, d\nu(y) \right) d\mu(x) = \int_{\mathbb{Y}} \left(\int_{\mathbb{X}} f^y(x)\, d\mu(x) \right) d\nu(y).$$

2. *Fubini: The same holds with* $L^+(\mathbb{X} \times \mathbb{Y})$ *replaced by* $L^1(\mu \times \nu)$*.*

Comment: Fubini's theorem states that under those conditions, in short-hand notation

$$\int_{\mathbb{X}\times\mathbb{Y}} f\, d(\mu \times \nu) = \int_{\mathbb{X}} \left(\int_{\mathbb{Y}} f\, d\nu \right) d\mu = \int_{\mathbb{Y}} \left(\int_{\mathbb{X}} f\, d\mu \right) d\nu.$$

We will commonly omit the brackets, and simply write

$$\int_{\mathbb{X}\times\mathbb{Y}} f\, d(\mu \times \nu) = \int_{\mathbb{X}} \int_{\mathbb{Y}} f\, d\nu d\mu = \int_{\mathbb{Y}} \int_{\mathbb{X}} f\, d\mu d\nu.$$

Proof: For $E \in \Sigma_{\mathbb{X}} \otimes \Sigma_{\mathbb{Y}}$ and $f = \chi_E$, the first part (Tonelli's theorem) coincides with the Theorem 3.52. By additivity, Tonelli's theorem holds for any simple function. If $f \in L^+(\mathbb{X} \times \mathbb{Y})$, let f_n be a sequence of simple functions increasing to f. Then, $(f_n)_x$ increases to f_n and $(f_n)^y$ increases to f^y; by Monotone Convergence,

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \int_{\mathbb{Y}} (f_n)_x(y)\, d\nu(y) = \int_{\mathbb{Y}} f_x(y)\, d\nu(y) \equiv g(x)$$

and

$$\lim_{n\to\infty} h_n(y) = \lim_{n\to\infty} \int_{\mathbb{X}} (f_n)^y(x)\, d\mu(x) = \int_{\mathbb{X}} f^y(x)\, d\mu(x) \equiv h(y),$$

hence g and h are in L^+. Finally, using once again Monotone Convergence,

$$\int_{\mathbb{X}} g\, d\mu = \lim_{n\to\infty} \int_{\mathbb{X}} g_n\, d\mu = \lim_{n\to\infty} \int_{\mathbb{X}\times\mathbb{Y}} f_n\, d(\mu \times \nu) = \int_{\mathbb{X}\times\mathbb{Y}} f\, d(\mu \times \nu),$$

and

$$\int_{\mathbb{Y}} h\,d\nu = \lim_{n\to\infty} \int_{\mathbb{Y}} h_n\,d\nu = \lim_{n\to\infty} \int_{\mathbb{X}\times\mathbb{Y}} f_n\,d(\mu\times\nu) = \int_{\mathbb{X}\times\mathbb{Y}} f\,d(\mu\times\nu).$$

This proves Tonelli's theorem.

Suppose now that

$$f \in L^+(\mathbb{X}\times\mathbb{Y}) \cap L^1(\mu\times\nu).$$

Then, Tonelli's theorem obviously holds and in particular, $g \in L^1(\mu)$ and $h \in L^1(\nu)$. Fubini's theorem follows by applying it to f^+ and f^- separately (and if f is complex-valued, to its real and imaginary parts). ■

A note on completeness: even if μ and ν are complete, the product measure is almost never complete. This is easy to see: suppose that $A \in \Sigma_{\mathbb{X}}$ satisfies $\mu(A) = 0$, ν is a finite measure and $\mathscr{P}(\mathbb{Y})$ contains non-measurable sets. Then, every set

$$E \in A \times (\mathscr{P}(\mathbb{Y}) \setminus \Sigma_{\mathbb{Y}})$$

is not in $\Sigma_{\mathbb{X}} \otimes \Sigma_{\mathbb{Y}}$ (otherwise E_x would be in $\Sigma_{\mathbb{Y}}$ for all x), and yet

$$E \subset A \times \mathbb{Y} \qquad \text{and} \qquad \mu\times\nu(A\times\mathbb{Y}) = 0.$$

Given two complete measures μ and ν we can consider the completion of $\mu\times\nu$. Fubini's theorem can be adjusted to this case, but note that measurability becomes then an issue.

✎ *Exercise 3.27* Consider the measure space $(\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2), m)$ and the set

$$E = \{(x,x) \,:\, x \in \mathbb{R}\}.$$

Prove that E is measurable and that $m(E) = 0$.

✎ *Exercise 3.28* Let $E \subset \mathbb{R}^2$ be Borel-measurable and let m be the standard Borel measure. Let

$$\tilde{E} = \{(ax, by) : (x,y) \in E\},$$

where $a, b \neq 0$. Prove that $\tilde{E}$ is measurable and that $m(\tilde{E}) = ab\,m(E)$.

✎ *Exercise 3.29* Let $(\mathbb{X}, \Sigma, \mu)$ be a complete measure space and let $f \in L^+(\mathbb{X}, \Sigma) \cap L^1(\mu)$. Prove that

$$\int_{\mathbb{X}} f\,d\mu = \int_{[0,\infty)} \mu(\{x \,:\, f(x) \geq t\})\,dm(t).$$

3.8 The n-dimensional Lebesgue integral

TA material 3.3 Prove that the Lebesgue measure in $\mathbb{R}^2$ is invariant under rigid transformations.