Chapter 4

Linear Forms

4.1 Definition and examples

Let V be a vector space over $\mathbb{F}$. Often, we want to assign vectors numerical values (think of measurements). In the context of a vector space over a field $\mathbb{F}$, the "number" we associate with each vector is a scalar; in other words, a "measurement" of vectors is a function $V \to \mathbb{F}$. However, a vector space is not just any old set of points; this set is endowed with an algebraic structure, and therefore, we may be interested in functions on V that "communicate" with this algebraic structure. This leads us to the following definition:

Definition 4.1 *Let V be a vector space over $\mathbb{F}$. A **linear form** (**תבנית לינארית**) or a **linear functional** (פונקציונל לינארי) over V is a function $\ell : V \to \mathbb{F}$ (i.e., a scalar-valued function with domain V) satisfying the following conditions: for every $\mathbf{u}, \mathbf{v} \in V$,*

$$\ell(\mathbf{u} + \mathbf{v}) = \ell(\mathbf{u}) + \ell(\mathbf{v}),$$

and for every $\mathbf{v} \in V$ and $a \in \mathbb{F}$,

$$\ell(a\,\mathbf{v}) = a\,\ell(\mathbf{v}).$$

In other words, a linear form on a vector space is a scalar-valued function over that space that "respects" linear operations. Note (once again) the distinction between operations in V and operations in $\mathbb{F}$.

Example: The function $\ell : V \to \mathbb{F}$ assigning to every vector $\mathbf{v} \in V$ the value $\ell(\mathbf{v}) = 0_{\mathbb{F}}$ is a linear form. Why? because for every $\mathbf{u}, \mathbf{v} \in V$ and $a \in \mathbb{F}$,

$$\ell(\mathbf{u} + \mathbf{v}) = 0_{\mathbb{F}} = 0_{\mathbb{F}} + 0_{\mathbb{F}} = \ell(\mathbf{u}) + \ell(\mathbf{v}),$$

and

$$\ell(a\,\mathbf{v}) = 0_{\mathbb{F}} = a\,\ell(\mathbf{v}).$$

This linear form is called the **zero form** (תבנית האפס). ▲▲▲

Example: Let V be an n-dimensional vector space and let

$$\mathfrak{B} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$$

be an ordered basis. For every $i = 1, \dots, n$, we denote by $\ell^i : V \to \mathbb{F}$ the function returning the i-th coordinate of a vector relative to the basis $\mathfrak{B}$. That is,

$$\ell^i(\mathbf{v}) = ([\mathbf{v}]_{\mathfrak{B}})^i.$$

More explicitly, if

$$\mathbf{v} = \begin{pmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{pmatrix} \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix},$$

then $\ell^i(\mathbf{v}) = a^i$. Why is this a linear form? Because for every $\mathbf{u}, \mathbf{v} \in V$,

$$\ell^i(\mathbf{u} + \mathbf{v}) = ([\mathbf{u} + \mathbf{v}]_{\mathfrak{B}})^i = ([\mathbf{u}]_{\mathfrak{B}} + [\mathbf{v}]_{\mathfrak{B}})^i = ([\mathbf{u}]_{\mathfrak{B}})^i + ([\mathbf{v}]_{\mathfrak{B}})^i = \ell^i(\mathbf{u}) + \ell^i(\mathbf{v}),$$

where we used here Proposition 3.46. Note the different types of addition: in the first two terms it is addition in V, in the third term it is addition in $\mathbb{F}^n_{\text{col}}$, and in the last two terms it is addition in $\mathbb{F}$.
Likewise, using once again Proposition 3.46, for $\mathbf{u} \in V$ and $c \in \mathbb{F}$,

$$\ell^i(c\,\mathbf{u}) = ([c\,\mathbf{u}]_{\mathfrak{B}})^i = (c\,[\mathbf{u}]_{\mathfrak{B}})^i = c\,([\mathbf{u}]_{\mathfrak{B}})^i = c\,\ell^i(\mathbf{u}),$$

Note that for every $i, j = 1, \dots, n$,

$$\ell^i(\mathbf{v}_j) = ([\mathbf{v}_j]_{\mathfrak{B}})^i = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases},$$

i.e., $\ell^i(\mathbf{v}_j) = \delta^i_j$. This particular set of linear forms will have an important role shortly. ▲▲▲

Example: Let $V = (\mathbb{F}^n_{\text{col}}, +, \mathbb{F}, \cdot)$ and let $\mathbf{a} \in \mathbb{F}^n_{\text{row}}$. We define the function $\ell_{\mathbf{a}} : V \to \mathbb{F}$ by

$$\ell_{\mathbf{a}}(\mathbf{v}) = \mathbf{a}\,\mathbf{v} = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}.$$

The function ℓ_a is a linear form because matrix multiplication is distributive, namely, for $\mathbf{u}, \mathbf{v} \in V$ and $c \in \mathbb{F}$,

$$\ell_{\mathbf{a}}(\mathbf{u} + \mathbf{v}) = \mathbf{a}\,(\mathbf{u} + \mathbf{v}) = \mathbf{a}\,\mathbf{u} + \mathbf{a}\,\mathbf{v} = \ell_{\mathbf{a}}(\mathbf{u}) + \ell_{\mathbf{a}}(\mathbf{v}),$$

and

$$\ell_{\mathbf{a}}(c\,\mathbf{u}) = \mathbf{a}\,(c\mathbf{u}) = c\,\mathbf{a}\,\mathbf{u} = c\,\ell_{\mathbf{a}}(\mathbf{u}).$$

Note how we view the row vector $\mathbf{a}$ as "constant" whereas the linear form $\ell_{\mathbf{a}}$ operates on all $\mathbf{v} \in V$. To summarize: every vector $\mathbf{a} \in \mathbb{F}^n_{\text{row}}$ defines via matrix multiplication a linear form on $\mathbb{F}^n_{\text{col}}$. ▲▲▲

Example: Take $n = 1$ and $\mathbb{F} = \mathbb{R}$ in the previous example; then $V = \mathbb{R}$, and for every $a \in \mathbb{R}$ we define the function

$$\ell_a(x) = a\,x.$$

Thus, linear forms coincide in this case with the good old notion of linear functions $\mathbb{R} \to \mathbb{R}$. ▲▲▲

Example: Let $V = (M_n(\mathbb{F}), +, \mathbb{F}, \cdot)$ and define the function known as the **trace** (עקבה) of the matrix.

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a^i_i.$$

It is readily verified that the trace is also a linear form. ▲▲▲

Example: Let S be a non-empty set (it doesn't need to have any other structure than being a set) and consider the set $V = \mathbb{F}^S$ of all functions $f : S \to \mathbb{F}$. We have seen that V is a vector space over $\mathbb{F}$ with respect to the natural operations of addition and scalar multiplication of field-valued functions (make sure you remember the vectorial structure of $\mathbb{F}^S$). Let $s \in S$, and define the function $\operatorname{Eval}_s : V \to \mathbb{F}$,

$$\operatorname{Eval}_s(f) = f(s).$$

(Given a function $f \in \mathbb{F}^S$, the function Eval_s return the value of f at s.) Then, Eval_s is a linear form, because for every $f, g \in \mathbb{F}^S$ and $c \in \mathbb{F}$,

$$\mathrm{Eval}_s(f+g) = (f+g)(s) = f(s) + g(s) = \mathrm{Eval}_s(f) + \mathrm{Eval}_s(g),$$

and

$$\mathrm{Eval}_s(c\,f) = (c\,f)(s) = c\,f(s) = c\,\mathrm{Eval}_s(f).$$

▲▲▲

4.2 Properties of linear forms

In this section we review some important properties of linear forms.

The following is readily proved inductively:

Proposition 4.2 *Let ℓ be a linear form on a vector space $(V, +, \mathbb{F}, \cdot)$. Then for every $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$ and $a^1, \dots, a^n \in \mathbb{F}$,*

$$\ell\left(a^1\mathbf{v}_1 + \cdots + a^n\mathbf{v}_n\right) = a^1\,\ell(\mathbf{v}_1) + \cdots + a^n\,\ell(\mathbf{v}_n).$$

Proof: This is left as an exercise. ■

Proposition 4.3 *Let ℓ be a linear form on a vector space $(V, +, \mathbb{F}, \cdot)$. Then*

$$\ell(0_V) = 0_{\mathbb{F}}.$$

Proof: Let $\mathbf{v} \in V$ be arbitrary. Then, using the fact that $0_{\mathbb{F}}\,\mathbf{v} = 0_V$ and the properties of ℓ,

$$\ell(0_V) = \ell(0_{\mathbb{F}}\,\mathbf{v}) = 0_{\mathbb{F}}\,\ell(\mathbf{v}) = 0_{\mathbb{F}}.$$

■

An important fact about linear forms (in finitely-generated vector spaces) is that they are completely determined by their action on basis vectors. We establish this in two separate propositions:

Proposition 4.4 *Let V be a finitely-generated vector space, and let*

$$\mathfrak{B} = \begin{pmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{pmatrix}$$

be an ordered basis for V. Then, for every set $c_1, \dots, c_n$ of scalars there exists a linear form ℓ, such that

$$\ell(\mathbf{v}_i) = c_i \qquad \text{for every } i = 1, \dots, n.$$

Proof: There really is only one way to define such a functional. Since every $\mathbf{v} \in V$ has a unique representation as

$$\mathbf{v} = a^1 \mathbf{v}_1 + \dots + a^n \mathbf{v}_n,$$

then $\ell(\mathbf{v})$ must be given by

$$\ell(\mathbf{v}) = a^1 \, \ell(\mathbf{v}_1) + \dots + a^n \, \ell(\mathbf{v}_n) = a^1 c_1 + \dots + a^n c_n.$$

To complete the proof, we have to verify that ℓ is a linear form. Let $\mathbf{v}, \mathbf{w} \in V$ be given by

$$\begin{aligned} \mathbf{v} &= a^1 \mathbf{v}_1 + \dots + a^n \mathbf{v}_n \\ \mathbf{w} &= b^1 \mathbf{v}_1 + \dots + b^n \mathbf{v}_n. \end{aligned}$$

Then,

$$\mathbf{v} + \mathbf{w} = (a^1 + b^1)\, \mathbf{v}_1 + \dots + (a^n + b^n)\, \mathbf{v}_n.$$

By the way we defined ℓ,

$$\begin{aligned} \ell(\mathbf{v}) &= a^1 \, c_1 + \dots + a^n \, c_n \\ \ell(\mathbf{w}) &= b^1 \, c_1 + \dots + b^n \, c_n, \end{aligned}$$

and

$$\ell(\mathbf{v} + \mathbf{w}) = (a^1 + b^1)\, c_1 + \dots + (a^n + b^n)\, c_n,$$

so that indeed $\ell(\mathbf{v} + \mathbf{w}) = \ell(\mathbf{v}) + \ell(\mathbf{w})$. We proceed similarly to show that $\ell(k\,\mathbf{v}) = k\,\ell(\mathbf{v})$ for $k \in \mathbb{F}$. ■

The following complementing proposition asserts that there really was no other way to define ℓ:

Proposition 4.5 *Let V be a finitely-generated vector space. Let*

$$\mathfrak{B} = \begin{pmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{pmatrix}$$

be an ordered basis for V. If two linear forms ℓ, ℓ' satisfy

$$\ell(\mathbf{v}_i) = \ell'(\mathbf{v}_i) \qquad \text{for all } i = 1, \dots, n,$$

then $\ell = \ell'$.

Proof: By the property of a basis in a finitely-generated vector space, every $\mathbf{v} \in V$ can be represented uniquely as

$$\mathbf{v} = a^1 \mathbf{v}_1 + \dots + a^n \mathbf{v}_n$$

for some scalars $a^1, \dots, a^n$. Then, by the linearity of ℓ, ℓ',

$$\ell(\mathbf{v}) = a^1 \ell(\mathbf{v}_1) + \dots + a^n \ell(\mathbf{v}_n) = a^1 \ell'(\mathbf{v}_1) + \dots + a^n \ell'(\mathbf{v}_n) = \ell'(\mathbf{v}).$$

■

Note how we defined the functional ℓ. Given the $\mathbf{c} \in \mathbb{F}^n_{\text{row}}$,

$$\ell(\mathbf{v}) = \begin{bmatrix} c_1 & \dots & c_n \end{bmatrix} \begin{bmatrix} ([\mathbf{v}]_{\mathfrak{B}})^1 \\ \vdots \\ ([\mathbf{v}]_{\mathfrak{B}})^n \end{bmatrix} = \mathbf{c}[\mathbf{v}]_{\mathfrak{B}}.$$

The two last propositions have a very important implication: every linear form can be defined using n scalars. It is difficult not to make a connection with the notion of coordinates. However, at this stage we haven't identified the set of linear forms as a vector space, hence these is yet no meaning to assign them coordinates. This will be rectified in the next section.

Take the particular example where $V = \mathbb{F}^n$ along with the standard basis,

$$\mathfrak{E} = \begin{pmatrix} \mathbf{e}_1 & \dots & \mathbf{e}_n \end{pmatrix}.$$

Then every vector $\mathbf{v} = (v^1, \dots, v^n) \in V$ "coincides with its coordinates", i.e., $v^i = ([\mathbf{v}]_{\mathfrak{B}})^i$. We have just shown that to every linear form ℓ corresponds a unique $\mathbf{c} \in \mathbb{F}^n_{\text{row}}$, such that

$$\ell(\mathbf{v}) = \mathbf{c}\,[\mathbf{v}]_{\mathfrak{E}} = c_1 v^1 + \dots + c_n v^n.$$

Exercises

(easy) 4.1 Prove using induction that for a linear from ℓ on a vector space V,

$$f(a^1\mathbf{v}_1 + \cdots + a^n\mathbf{v}_n) = a^1 f(\mathbf{v}_1) + \ldots a^n f(\mathbf{v}_n)$$

for every $a^1, \ldots, a^n \in \mathbb{F}$ and $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$.

(intermediate) 4.2 Let $V = (\mathbb{R}^3, +, \mathbb{R}, \cdot)$ and let

$$\mathbf{v}_1 = (1,0,1) \qquad \mathbf{v}_2 = (0,1,-2) \qquad \text{and} \qquad \mathbf{v}_3 = (-1,-1,0).$$

(a) Find the linear form ℓ on $\mathbb{R}^3$ satisfying

$$\ell(\mathbf{v}_1) = 1 \qquad \ell(\mathbf{v}_2) = -2 \qquad \text{and} \qquad \ell(\mathbf{v}_3) = 3.$$

That is, what is $\ell(x,y,z)$?

(b) Characterize all linear forms satisfying $\ell(\mathbf{v}_1) = \ell(\mathbf{v}_2) = 0$ and $\ell(\mathbf{v}_3) \neq 0$.

(c) Show that for a linear form such as in the previous article, $\ell(2,3,-1) \neq 0$.

(intermediate) 4.3 Let $(V, +, \mathbb{F}, \cdot)$ be a finitely-generated vector space and let $\mathbf{v} \in V$ be a non-zero vector, $\mathbf{v} \neq 0_V$. Prove that there exists a linear form $\ell \in V^\vee$, such that $\ell(\mathbf{v}) \neq 0_\mathbb{F}$.

(intermediate) 4.4 Let $(V, +, \mathbb{F}, \cdot)$ be a finitely-generated vector space and let $\mathbf{u}, \mathbf{v} \in V$ be distinct vectors, $\mathbf{u} \neq \mathbf{v}$. Prove that there exists a linear form $\ell \in V^\vee$, such that $\ell(\mathbf{u}) \neq \ell(\mathbf{v})$.

(intermediate) 4.5 Let $(V, +, \mathbb{F}, \cdot)$ be a vector space and let $\ell, m \in V^\vee$ be linear forms satisfying that

$$\ell(\mathbf{v}) = 0_\mathbb{F} \qquad \text{if and only if} \qquad m(\mathbf{v}) = 0_\mathbb{F}.$$

Prove that there exists an $a \in \mathbb{F}$ such that $m = a\,\ell$.

(intermediate) 4.6 Consider the infinite-dimensional vector space $\mathbb{R}[X]$. Let $a, b \in \mathbb{R}$ such that $a < b$. For

$$P = \sum_{i=0}^{n} p_i X^i \in \mathbb{R}[X]$$

we define

$$\int_a^b P(x)\,dx = \sum_{i=0}^{n} \frac{p_i}{i+1}(b^{i+1} - a^{i+1}).$$

Let $Q \in \mathbb{R}[X]$. Prove that the function $\ell : \mathbb{R}[X] \to \mathbb{R}$ defined by

$$\ell(P) = \int_a^b P(x)Q(x)\,dx$$

is a linear form. Note: you are not expected to know anything about integrals—just follow the definitions.

4.3 The dual space

Let V be a vector space over $\mathbb{F}$. In the previous section we defined the notion of linear forms over $(V, +, \mathbb{F}, \cdot)$. We denote the set of all linear forms over V by

$$V^\vee = \{\ell : V \to \mathbb{F} \;:\; \ell \text{ is a linear form}\}.$$

it is a subset of the set of $\mathrm{Func}(V, \mathbb{F})$, which comprises all (i.e., not necessarily linear) functions $f : V \to \mathbb{F}$. Recall that $\mathrm{Func}(V, \mathbb{F})$ is itself a vector space over $\mathbb{F}$ with respect to the function addition

$$(f + g)(\mathbf{v}) = f(\mathbf{v}) + g(\mathbf{v})$$

and the scalar multiplication

$$(c\,f)(\mathbf{v}) = c\,f(\mathbf{v}).$$

Proposition 4.6 *The set of linear forms $V^\vee$ is a linear subspace of the vector space* $\mathrm{Func}(V, \mathbb{F})$ *(hence, $V^\vee$ is a vector space in its own sake).*

Proof: By definition, in order to prove that a set of vectors is a linear subspace, we need to prove that it is non-empty, and that it is closed under addition and scalar multiplication.

The set $V^\vee$ is non-empty, because it contains at least the zero form, which we now denote by $0_{V^\vee}$. Let $\ell_1, \ell_2 \in V^\vee$. The sum $\ell_1 + \ell_2$ is well-defined as a sum in $\text{Func}(V, \mathbb{F})$; we need to show that $\ell_1 + \ell_2 \in V^\vee$, i.e., that it is a linear form. For all $\mathbf{u}, \mathbf{v} \in V$ and $c \in \mathbb{F}$,

$$\begin{aligned}(\ell_1 + \ell_2)(\mathbf{u} + \mathbf{v}) &= \ell_1(\mathbf{u} + \mathbf{v}) + \ell_2(\mathbf{u} + \mathbf{v}) \\ &= (\ell_1(\mathbf{u}) + \ell_1(\mathbf{v})) + (\ell_2(\mathbf{u}) + \ell_2(\mathbf{v})) \\ &= (\ell_1(\mathbf{u}) + \ell_2(\mathbf{u})) + (\ell_1(\mathbf{v}) + \ell_2(\mathbf{v})) \\ &= (\ell_1 + \ell_2)(\mathbf{u}) + (\ell_1 + \ell_2)(\mathbf{v}),\end{aligned}$$

and

$$\begin{aligned}(\ell_1 + \ell_2)(c\,\mathbf{u}) &= \ell_1(c\,\mathbf{u}) + \ell_2(c\,\mathbf{u}) \\ &= c\,\ell_1(\mathbf{u}) + c\,\ell_2(\mathbf{u}) \\ &= c\,(\ell_1(\mathbf{u}) + \ell_2(\mathbf{u})) \\ &= c\,(\ell_1 + \ell_2)(\mathbf{u}),\end{aligned}$$

proving that $\ell_1 + \ell_2 \in V^\vee$. Likewise, let $\ell \in V^\vee$ and $a \in \mathbb{F}$; we need to show that $a\,\ell \in V^\vee$, i.e., that it is a linear form. For all $\mathbf{u}, \mathbf{v} \in V$ and $c \in \mathbb{F}$,

$$\begin{aligned}(a\,\ell)(\mathbf{u} + \mathbf{v}) &= a\,\ell(\mathbf{u} + \mathbf{v}) \\ &= a\,(\ell(\mathbf{u}) + \ell(\mathbf{v})) \\ &= a\,\ell(\mathbf{u}) + a\,\ell(\mathbf{v}) \\ &= (a\,\ell)(\mathbf{u}) + (a\,\ell)(\mathbf{v}),\end{aligned}$$

and

$$\begin{aligned}(a\,\ell)(c\,\mathbf{u}) &= a\,\ell(c\,\mathbf{u}) \\ &= a\,(c\,\ell(\mathbf{u})) \\ &= c\,(a\,\ell(\mathbf{u})) \\ &= c\,(a\,\ell)(\mathbf{u}),\end{aligned}$$

proving that $a\,\ell \in V^\vee$. This completes the proof. ■

Thus, every vector space $(V, +, \mathbb{F}, \cdot)$ induces another vector space $(V^\vee, +, \mathbb{F}, \cdot)$ over the same field. The vector space $V^\vee$ is called the space **dual** (**דואלי**) to V. You should internalize the fact that elements of $V^\vee$ are also vectors, but they are at the same time functions over a vector space, V. Elements of V and elements of $V^\vee$ are both vectors, albeit belonging to different spaces. In particular, there is no meaning to adding an element of V and an element of

$V^\vee$. On the other hand, the elements of $V^\vee$ "act" on element of V to yield scalars.

The action $\ell(\mathbf{v})$ of a linear form ℓ or a vector $\mathbf{v}$ can be viewed as a function taking an element of $V^\vee$ and an element of V and returning is a scalar. We often denote this pairing by

$$\langle \cdot, \cdot \rangle : V^\vee \times V \to F,$$

where

$$\langle \ell, \mathbf{v} \rangle = \ell(\mathbf{v}).$$

Example: For $V = \mathbb{F}^n_{\text{col}}$ we have seen that $V^\vee$ can be identified with $\mathbb{F}^n_{\text{row}}$: every $\mathbf{a} \in \mathbb{F}^n_{\text{row}}$ defined a unique $\ell_{\mathbf{a}} \in V^\vee$ defined by

$$\ell_{\mathbf{a}}(\mathbf{v}) = \mathbf{a} \cdot \mathbf{v}.$$

It is customary to write

$$(\mathbb{F}^n_{\text{col}})^\vee \simeq \mathbb{F}^n_{\text{row}},$$

where the $\simeq$ sign mean that the two spaces can be identified (more on that later). ▲▲▲

4.4 Dual bases

Let V be a finitely-generated vector space. What can be said about its dual space? Is it also finitely-generated? If it is, is there a relation between $\dim_{\mathbb{F}} V$ and $\dim_{\mathbb{F}} V^\vee$? The theorem below answers this question affirmatively.

Theorem 4.7 *Let V be a finitely-generated vector space. Let*

$$\mathfrak{B} = \begin{pmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{pmatrix}$$

be an ordered basis for V. Then,

$$\mathfrak{B}^\vee = \begin{pmatrix} \ell^1 \\ \vdots \\ \ell^n \end{pmatrix}$$

is an ordered basis for $V^\vee$, called the ***dual basis*** *(בסיס דואלי) of $\mathfrak{B}$, where ℓ^i is the unique linear form satisfying*

$$\ell^i(\mathbf{v}_j) = \delta^i_j \qquad \text{for all } i,j = 1,\dots,n,$$

or equivalently

$$\ell^i(\mathbf{v}) = ([\mathbf{v}]_{\mathfrak{B}})^i.$$

As a result,

$$\dim_{\mathbb{F}} V^\vee = \dim_{\mathbb{F}} V.$$

Proof: We need to show that $\mathfrak{B}^\vee$ is spanning and independent. Suppose that $a_1,\dots,a_n$ are scalars satisfying

$$a_1\ell^1 + \cdots + a_n\ell^n = 0_{V^\vee}$$

(this is an equality between elements in $V^\vee$). In particular, applying both sides on $\mathbf{v}_j$,

$$a_1\ell^1(\mathbf{v}_j) + \cdots + a_n\,\ell^n(\mathbf{v}_j) = 0_{V^\vee}(\mathbf{v}_j) = 0_{\mathbb{F}},$$

i.e.,

$$a_j = 0_{\mathbb{F}}.$$

Since this holds for every $j = 1,\dots,n$, it follows that the linear combination of the ℓ^i's is trivial, namely, the linear forms ℓ^i are linearly-independent.

It remains to show that $\mathfrak{B}^\vee$ is spanning. We will show that any $\ell \in V^\vee$ can be represented as

$$\ell = \ell(\mathbf{v}_1)\,\ell^1 + \cdots + \ell(\mathbf{v}_n)\,\ell^n,$$

i.e., it is a linear combination of the linear forms ℓ^i (note that $\ell(\mathbf{v}_i)$ are scalars). By Proposition 4.5 it suffices to verify that both sides yield the same scalar when acting on basis vectors $\mathbf{v}_j$. Indeed,

$$(\ell(\mathbf{v}_1)\,\ell^1 + \cdots + \ell(\mathbf{v}_n)\,\ell^n)(\mathbf{v}_j) = \ell(\mathbf{v}_1)\,\ell^1(\mathbf{v}_j) + \cdots + \ell(\mathbf{v}_n)\,\ell^n(\mathbf{v}_j) = \ell(\mathbf{v}_j),$$

which completes the proof. ■

Example: Let $V = (\mathbb{F}^n, +, \mathbb{F}, \cdot)$ and let

$$\mathfrak{E} = \begin{pmatrix} \mathbf{e}_1 & \dots & \mathbf{e}_n \end{pmatrix}$$

be the standard basis. We denote the basis dual to $\mathfrak{E}$ by

$$\mathfrak{E}^\vee = \begin{pmatrix} \mathbf{e}^1 \\ \vdots \\ \mathbf{e}^n \end{pmatrix}.$$

As we have seen, for $\mathbf{v} = (x^1, \dots, x^n)$ we have

$$\mathbf{e}^i(\mathbf{v}) = [\mathbf{v}]_{\mathfrak{E}} = x^i,$$

that is the i-th linear form in the dual standard basis extracts the i-th coordinate of a vector. ▲▲▲

Since $V^\vee$ is a vector space and since $\mathfrak{B}^\vee$ is a basis for $V^\vee$, every linear form in $V^\vee$ can be represented using coordinates. Every $\ell \in V^\vee$ has a unique representation

$$\ell = \underbrace{\begin{bmatrix} c_1 & \dots & c_n \end{bmatrix}}_{[\ell]_{\mathfrak{B}^\vee}} \underbrace{\begin{pmatrix} \ell^1 \\ \vdots \\ \ell^n \end{pmatrix}}_{\mathfrak{B}^\vee},$$

where $[\ell]_{\mathfrak{B}^\vee} \in \mathbb{F}^n_{\text{row}}$ is the coordinate matrix. We have just proved that

$$[\ell]_{\mathfrak{B}^\vee} = \begin{bmatrix} \ell(\mathbf{v}_1) & \dots & \ell(\mathbf{v}_n) \end{bmatrix}.$$

Consider now the following question: given a basis $\mathfrak{B}$ on a finitely-generated vector space V, and its dual basis, every vector v and every linear form ℓ can be written using coordinates,

$$\mathbf{v} = \mathfrak{B}\,[\mathbf{v}]_{\mathfrak{B}} \qquad \text{and} \qquad \ell = [\ell]_{\mathfrak{B}^\vee}\,\mathfrak{B}^\vee.$$

Can we express the scalar $\ell(\mathbf{v})$ obtained by the action of the linear form on the vector using their respective coordinates?

Let denote the coordinates of $\mathbf{v}$ and ℓ as

$$\begin{aligned} \mathbf{v} &= a^1\mathbf{v}_1 + \dots + a^n\mathbf{v}_n \\ \ell &= b_1\ell^1 + \dots + b_n\ell^n, \end{aligned}$$

namely,

$$[\mathbf{v}]_{\mathfrak{B}} = \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix} \qquad \text{and} \qquad [\ell]_{\mathfrak{B}^\vee} = \begin{bmatrix} b_1 & \dots & b_n \end{bmatrix}.$$

Then,

$$\begin{aligned}\ell(\mathbf{v}) &= \sum_{i=1}^{n} b_i \ell^i \left(\sum_{j=1}^{n} a^j \mathbf{v}_j \right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} b_i a^j \ell^i(\mathbf{v}_j) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} b_i a^j \delta^i_j \\ &= \sum_{i=1}^{n} b_i a^i.\end{aligned}$$

Consider the right-hand side; it is the product of the row vector $[\ell]_{\mathfrak{B}^\vee}$ and the column vector $[v]_{\mathfrak{B}}$.

We have just proved the following:

Proposition 4.8 *Let V be a finitely-generated vector space. Let*

$$\mathfrak{B} = \begin{pmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{pmatrix}$$

be an ordered basis for V and let

$$\mathfrak{B}^\vee = \begin{pmatrix} \ell^1 \\ \vdots \\ \ell^n \end{pmatrix}$$

be its dual basis. Then, for every $\ell \in V^\vee$ and $\mathbf{v} \in V$,

$$\ell(\mathbf{v}) = [\ell]_{\mathfrak{B}^\vee} [\mathbf{v}]_{\mathfrak{B}}.$$

We have seen that given an ordered basis $\mathfrak{B} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ and its dual $\mathfrak{B}^\vee = (\ell^1, \dots, \ell^n)$ in a finitely-generated vector space, every linear form $\ell \in V^\vee$ can be represented as

$$\ell = \sum_{i=1}^{n} \ell(\mathbf{v}_i)\, \ell^i.$$

This representation has an analog for vectors: every vector $\mathbf{v} \in V$ is given by

$$\mathbf{v} = \sum_{i=1}^{n} \ell^i(\mathbf{v})\, \mathbf{v}_i,$$

because by definition, $\ell^i(\mathbf{v}) = ([\mathbf{v}]_{\mathfrak{B}})^i$.

We end this section with addressing the transition between dual bases:

Theorem 4.9 *Let V be a finitely-generated vector space. Let*

$$\mathfrak{B} = \begin{pmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{pmatrix} \qquad \text{and} \qquad \mathfrak{C} = \begin{pmatrix} \mathbf{w}_1 & \dots & \mathbf{w}_n \end{pmatrix}$$

be ordered bases for V, related by a transition matrix $P \in \mathrm{GL}_n(\mathbb{F})$,

$$\mathfrak{C} = \mathfrak{B}\, P.$$

Denote the corresponding dual bases by

$$\mathfrak{B}^\vee = \begin{pmatrix} \ell^1 \\ \vdots \\ \ell^n \end{pmatrix} \qquad \text{and} \qquad \mathfrak{C}^\vee = \begin{pmatrix} m^1 \\ \vdots \\ m^n \end{pmatrix}.$$

Then, the transition matrix from $\mathfrak{B}^\vee$ to $\mathfrak{C}^\vee$ is $Q = P^{-1}$,

$$\mathfrak{C}^\vee = Q\mathfrak{B}^\vee.$$

Proof: By definition of the dual basis,

$$m^j(\mathbf{w}_i) = \delta_i^j \qquad \text{for all } i, j = 1, \dots, n.$$

It is given that

$$\mathbf{w}_i = \sum_{k=1}^{n} p_i^k \mathbf{v}_k,$$

and we need to show that

$$m^j = \sum_{s=1}^{n} q_s^j \ell^s.$$

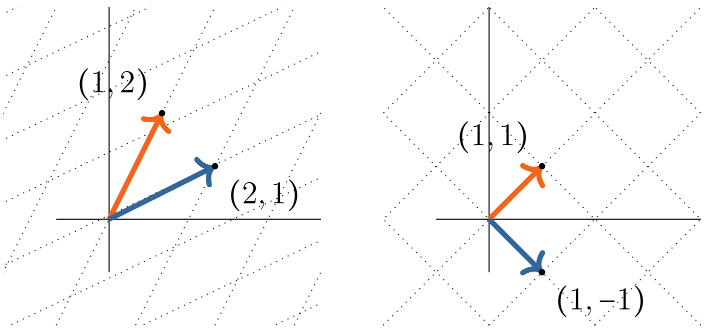
This is an identity between linear forms; both sides are equal if they yield the same set of scalars when acting on the basis vectors $\mathbf{w}_i$. Indeed, for every $i, j = 1, \dots, n$,

$$\begin{aligned}
\sum_{s=1}^{n} q_s^j \ell^s(\mathbf{w}_i) &= \sum_{s=1}^{n} q_s^j \ell^s\left(\sum_{k=1}^{n} p_i^k \mathbf{v}_k\right) \\
&= \sum_{s=1}^{n} q_s^j \sum_{k=1}^{n} p_i^k \ell^s(\mathbf{v}_k) \\
&= \sum_{s=1}^{n} q_s^j \sum_{k=1}^{n} p_i^k \delta_k^s \\
&= \sum_{k=1}^{n} q_k^j p_i^k \\
&= (PQ)_i^j = \delta_i^j.
\end{aligned}$$

This completes the proof. ■

Example: Consider once again the vector space $(\mathbb{R}^2, +, \mathbb{R}, \cdot)$ endowed with the two bases

$$\mathfrak{B} = \big((1,2) \quad (2,1)\big) \qquad \text{and} \qquad \mathfrak{C} = \big((1,1) \quad (1,-1)\big).$$



We have seen that

$$\underbrace{\big((1,1) \quad (1,-1)\big)}_{\mathfrak{C}} = \underbrace{\big((1,2) \quad (2,1)\big)}_{\mathfrak{B}} \underbrace{\begin{bmatrix} 1/3 & -1 \\ 1/3 & 1 \end{bmatrix}}_{P},$$

and

$$\underbrace{\big((1,2) \quad (2,1)\big)}_{\mathfrak{B}} = \underbrace{\big((1,1) \quad (1,-1)\big)}_{\mathfrak{C}} \underbrace{\begin{bmatrix} 3/2 & 3/2 \\ -1/2 & 1/2 \end{bmatrix}}_{Q}.$$

We now calculate the dual bases

$$\mathfrak{B}^\vee = \begin{pmatrix} \ell^1 \\ \ell^2 \end{pmatrix} \qquad \text{and} \qquad \mathfrak{C}^\vee = \begin{pmatrix} m^1 \\ m^2 \end{pmatrix}.$$

Since

$$\ell^i(\mathbf{v}) = ([\mathbf{v}]_{\mathfrak{B}})^i,$$

we have to find the coordinates of every vector $v \in \mathbb{R}^2$ relative to the basis $\mathfrak{B}$. Write $\mathbf{v} = (x,y)$, then

$$(x,y) = \ell^1(\mathbf{v})(1,2) + \ell^2(\mathbf{v})(2,1),$$

from which we obtain that

$$\ell^1(x,y) = \tfrac{1}{3}(2y - x) \qquad \text{and} \qquad \ell^2(x,y) = \tfrac{1}{3}(2x - y).$$

Similarly,

$$(x,y) = m^1(\mathbf{v})(1,1) + m^2(\mathbf{v})(1,-1),$$

from which we obtain that

$$m^1(x,y) = \tfrac{1}{2}(x + y) \qquad \text{and} \qquad m^2(x,y) = \tfrac{1}{2}(x - y).$$

Since $\mathfrak{C} = \mathfrak{B}P$ we expect that $\mathfrak{C}^\vee = Q\,\mathfrak{B}^\vee$, i.e.,

$$\begin{pmatrix} m^1 \\ m^2 \end{pmatrix} = \begin{bmatrix} 3/2 & 3/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{pmatrix} \ell^1 \\ \ell^2 \end{pmatrix}$$

Indeed, for every $\mathbf{v} = (x,y)$,

$$(\tfrac{3}{2}\,\ell^1 + \tfrac{3}{2}\,\ell^2)(\mathbf{v}) = \tfrac{3}{2} \cdot \tfrac{1}{3}(2y - x) + \tfrac{3}{2} \cdot \tfrac{1}{3}(2x - y) = \tfrac{1}{2}(x + y) = m^1(\mathbf{v}),$$

and

$$(-\tfrac{1}{2}\ell^1 + \tfrac{1}{2}\,\ell^2)(\mathbf{v}) = -\tfrac{1}{2} \cdot \tfrac{1}{3}(2y - x) + \tfrac{1}{2} \cdot \tfrac{1}{3}(2x - y) = \tfrac{1}{2}(x - y) = m^2(\mathbf{v}).$$

▲ ▲ ▲

Exercises

(easy) 4.7 Consider the vector space $(\mathbb{R}^2, +, \mathbb{R}, \cdot)$. Find the ordered basis dual to the ordered basis

$$\mathfrak{B} = \big((3,4) \quad (5,7)\big).$$

(intermediate) 4.8 Let $(V, +, \mathbb{F}, \cdot)$ be a finitely-generated vector space. Prove that

(a) $\mathbf{v} = 0_V$ if and only if $\ell(\mathbf{v}) = 0$ for all $\ell \in V^\vee$.

(b) $\ell = 0_{V^\vee}$ if and only if $\ell(\mathbf{v}) = 0$ for all $\mathbf{v} \in V$.

(intermediate) 4.9 Consider the vector space $(\mathbb{C}^3, +, \mathbb{C}, \cdot)$. Find the basis dual to the ordered basis

$$\mathfrak{B} = \big((1,0,-1) \quad (1,1,1) \quad (2,2,0)\big).$$

(intermediate) 4.10 Let $V = (\mathbb{Q}^3, +, \mathbb{Q}, \cdot)$ and consider the ordered basis

$$\mathfrak{B} = ((1,0,-1),(1,1,1),(2,2,0)).$$

(a) Find the basis $\mathfrak{B}^\vee$ dual to $\mathfrak{B}$.

(b) Let $\mathcal{E} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be the standard basis for V. Find the basis $\mathcal{E}^\vee$ dual to $\mathcal{E}$

(c) Find the transition matrix P satisfying $\mathfrak{B} = \mathcal{E}P$.

(d) Find the transition matrix Q satisfying $\mathcal{E}^\vee = Q\mathfrak{B}^\vee$ (write the bases $\mathcal{E}^\vee$ and $\mathfrak{B}^\vee$ as columns of linear forms).

(e) Find the transition matrix P satisfying $\mathcal{E} = \mathfrak{B}P$.

(f) Find the transition matrix Q satisfying $\mathfrak{B}^\vee = Q\mathcal{E}^\vee$.

(intermediate) 4.11 Repeat the previous question with $\mathcal{E}$ replaced by

$$\mathfrak{C} = ((1,1,0),(1,0,1),(0,1,1)).$$

(intermediate) 4.12 Based on the last two questions, formulate a general statement and prove it.

(intermediate) 4.13 Let $(V,+,\mathbb{F},\cdot)$ be a vector space of dimension *at least* n. Let $A \in \mathrm{GL}_n(\mathbb{F})$ (an invertible square matrix) and let

$$\begin{pmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{pmatrix}$$

be an independent sequence of vectors. Define the linear forms

$$\begin{pmatrix} \varphi^1 \\ \vdots \\ \varphi^n \end{pmatrix}$$

via

$$\varphi^i(\mathbf{v}_j) = a^i_j \qquad \text{for all } i,j = 1,\dots,n.$$

(Recall that this defines the linear forms uniquely.) Show that the linear forms $\varphi^1,\dots,\varphi^n$ are linearly-independent. Try to relate this question to the last three.

(harder) 4.14 Let $\mathfrak{B} = (\mathbf{v}_1,\mathbf{v}_2,\dots)$ be an infinite (but countable) basis for a vector space V over a field $\mathbb{F}$. Define a sequence of linear forms $\mathfrak{B}^\vee = (\ell^1,\ell^2,\dots)$ by

$$\ell^i(\mathbf{v}_j) = \delta^i_j.$$

(a) Show that the functions ℓ^i are indeed well-defined for all $\mathbf{v} \in V$, and are linear forms.

(b) Show that the sequence $\mathfrak{B}^\vee$ is linearly-independent.

(c) Show that $\mathfrak{B}^\vee$ is *not* a basis for $V^\vee$. I.e., there exists an $\ell \in V^\vee$ which is not in the span of $\mathfrak{B}^\vee$. Hint: set $\ell(\mathbf{v}_i) = 1$ for all $i \in \mathbb{N}$.

4.5 Null space and annihilator

4.5.1 The annihilator of a set of vectors

Definition 4.10 *Let V be a vector space over $\mathbb{F}$ and let $S \subseteq V$ be a subset (not necessarily a subspace). The* ***annihilator*** *(קבוצת המאפסים) of S is the set $S^0 \subseteq V^\vee$ of linear forms that vanish on all elements in S,*

$$S^0 = \{\ell \in V^\vee \ :\ \ell(\mathbf{v}) = 0_{\mathbb{F}} \text{ for all } \mathbf{v} \in S\} \subseteq V^\vee.$$

(In some places the notation is $\mathrm{Ann}(S)$*.)*

Example: Let $S = \{0_V\}$, then the set of linear forms $\ell \in V^\vee$ satisfying that $\ell(\mathbf{v}) = 0_\mathbb{F}$ for all $\mathbf{v} \in S$, i.e., $\ell(0_V) = 0_\mathbb{F}$ is the entirety of $V^\vee$, i.e.,

$$\{0_V\}^0 = V^\vee.$$

▲▲▲

Example: Let $V = (\mathbb{R}^2, +, \mathbb{R}, \cdot)$ and let $S = \{(1,0)\}$. Then,

$$S^0 = \{\ell \in V^\vee \ : \ \ell(1,0) = 0_\mathbb{F}\}.$$

Take the standard basis for $V^\vee$,

$$\mathbf{e}^1(x,y) = x \qquad \text{and} \qquad \mathbf{e}^2(x,y) = y.$$

Writing $\ell = a\,\mathbf{e}^1 + b\,\mathbf{e}^2$, we have that

$$\ell(1,0) = 0_\mathbb{F} \qquad \text{if and only if} \qquad a = 0_\mathbb{F},$$

so that

$$S^0 = \{b\,\mathbf{e}^2 \ : \ b \in \mathbb{F}\} = \mathbb{F}\,\mathbf{e}^2.$$

▲▲▲

Example: Let $V = (\mathbb{R}^2, +, \mathbb{R}, \cdot)$ and let $S = \{(1,0), (0,1)\}$. Then,

$$S^0 = \{\ell \in V^\vee \ : \ \ell(1,0) = 0_\mathbb{F} \quad \text{and} \quad \ell(0,1) = 0_\mathbb{F}\}.$$

Using the same basis for $V^\vee$, we obtain that both a and b vanish, i.e.,

$$S^0 = \{0_{V^\vee}\}.$$

▲▲▲

Look at the above three example: first notice that the larger S is, the smaller S^0 is. Second, in all instances S^0 turned out to be a linear subspace of $V^\vee$. The next two propositions show that this is always the case:

Proposition 4.11 *Let V be a vector space over $\mathbb{F}$ and let $S \subseteq V$ be a subset. Then,*

$$S^0 \le V^\vee.$$

Proof: We need to show that S^0 is non-empty and that it is closed under addition and scalar multiplication. The set S^0 is non-empty because $0_{V^\vee} \in S^0$. Let $\ell, \ell' \in S^0$, i.e.,

$$\ell(\mathbf{v}) = \ell'(\mathbf{v}) = 0_{\mathbb{F}} \qquad \text{for all } \mathbf{v} \in S.$$

Then,

$$(\ell + \ell')(\mathbf{v}) = \ell(\mathbf{v}) + \ell'(\mathbf{v}) = 0_{\mathbb{F}} \qquad \text{for all } \mathbf{v} \in S,$$

proving that $\ell + \ell' \in S^0$. Likewise, let $\ell \in S^0$ and $a \in \mathbb{F}$, then

$$(a\,\ell)(\mathbf{v}) = a\,\ell(\mathbf{v}) = 0_{\mathbb{F}} \qquad \text{for all } \mathbf{v} \in S,$$

proving that $a\,\ell \in S^0$. By definition, $S^0 \le V^\vee$. ■

Proposition 4.12 *Let* $(V, +, \mathbb{F}, \cdot)$ *be a vector space and let* $S, T \subseteq V$. *Then,*

(a) *If* $S \subseteq T$ *then* $T^0 \le S^0$.
(b) $S^0 = (\operatorname{Span} S)^0$

Proof: For the first item, let $\ell \in T^0$, i.e.,

$$\ell(\mathbf{v}) = 0_{\mathbb{F}} \qquad \text{for all } \mathbf{v} \in T.$$

Since $S \subseteq T$, it follows that

$$\ell(\mathbf{v}) = 0_{\mathbb{F}} \qquad \text{for all } \mathbf{v} \in S,$$

i.e., $\ell \in S^0$, proving that $T^0 \subseteq S^0$.
For the second item, let $\ell \in S^0$, i.e.,

$$\ell(\mathbf{v}) = 0_{\mathbb{F}} \qquad \text{for all } \mathbf{v} \in S.$$

Every $\mathbf{v} \in \operatorname{Span} S$ is of the form

$$\mathbf{v} = a^1\mathbf{v}_1 + \cdots + a^n\mathbf{v}_n$$

for some $\mathbf{v}_1, \dots, \mathbf{v}_n \in S$, hence

$$\ell(\mathbf{v}) = a^1\,\ell(\mathbf{v}_1) + \cdots + a^n\,\ell(\mathbf{v}_n) = 0_{\mathbb{F}},$$

proving that $\ell \in (\operatorname{Span} S)^0$, i.e.,

$$S^0 \subseteq (\operatorname{Span} S)^0.$$

Conversely, since $S \subseteq \operatorname{Span} S$, it follows from the first item that $(\operatorname{Span} S)^0 \subseteq S^0$, proving that $(\operatorname{Span} S)^0 = S^0$. ■

Thus far, S was just any old set; consider now the case there $S = W$ is a subspace of V, in which case we have two subspaces, W and W^0, of spaces, V and $V^\vee$, having the same dimension. As we show the dimensions of W and W^0 are inter-related:

Proposition 4.13 *Let* $(V, +, \mathbb{F}, \cdot)$ *be a finitely-generated vector space and let* $W \leq V$*. Then,*

$$\dim_{\mathbb{F}} W + \dim_{\mathbb{F}} W^0 = \dim_{\mathbb{F}} V.$$

Proof: Suppose that

$$\dim_{\mathbb{F}} W = n \qquad \text{and} \qquad \dim_{\mathbb{F}} V = n + k.$$

Let $\begin{pmatrix}\mathbf{w}_1 & \dots & \mathbf{w}_n\end{pmatrix}$ be an ordered basis for W, which we complete (using Proposition 3.36) into an ordered basis

$$\mathfrak{B} = (\mathbf{w}_1, \dots, \mathbf{w}_n, \mathbf{v}_1, \dots, \mathbf{v}_k)$$

for V. We partition its dual basis accordingly

$$\mathfrak{B}^\vee = (\ell^1, \dots, \ell^n, m^1, \dots, m^k),$$

such that

$$\ell^i(\mathbf{w}_j) = \delta^i_j \qquad \ell^i(\mathbf{v}_j) = 0 \qquad m^i(\mathbf{w}_j) = 0 \quad \text{and} \quad m^i(\mathbf{v}_j) = \delta^i_j.$$

We will be done if we prove that $(m_1, \dots, m_k)$ is an ordered basis for W^0, for then $\dim_{\mathbb{F}} W^0 = k$.

By the definition of a basis, every $\ell \in W^0 \le V^\vee$ can be written as

$$\ell = (a_1\ell^1 + \cdots + a_n\ell^n) + (b_1 m^1 + \cdots + b_k m^k).$$

For every $j = 1, \dots, n$,

$$0_{\mathbb{F}} = \ell(\mathbf{w}_j) = (a_1\ell^1 + \cdots + a_n\ell^n)(\mathbf{w}_j) + (b_1 m^1 + \cdots + b_k m^k)(\mathbf{w}_j) = a_j,$$

proving that

$$\ell = b_1 m^1 + \cdots + b_k m^k,$$

i.e., $(m^1, \dots, m^k)$ is a generating set for W^0; since it is also independent, it is a basis for W^0. ■

4.5.2 The null space of a set of linear forms

The notion of an annihilating set has a dual version:

Definition 4.14 *Let V be vector space and let $L \subseteq V^\vee$. The **null space** (קבוצת האפסים) of L is the set of vectors*

$$L_0 = \{\mathbf{v} \in V \;:\; \ell(\mathbf{v}) = 0_{\mathbb{F}} \text{ for all } \ell \in L\} \subseteq V.$$

Example: Let V be any vector space and $L = \{0_{V^\vee}\}$. Then,

$$L_0 = \{\mathbf{v} \in V \;:\; 0_{V^\vee}(\mathbf{v}) = 0_{\mathbb{F}}\} = V.$$

▲▲▲

Example: Let $V = \mathbb{F}^3_{\text{col}}$ and let $L = \{\ell\}$ for

$$\ell([x, y, z]^T) = x + y + z.$$

Then,

$$L_0 = \{([x, y, z]^T) \in \mathbb{F}^3_{\text{col}} \;:\; x + y + z = 0\},$$

which we know how to express explicitly. In fact, we know that

$$L_0 = \left[\begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} \;:\; s, t \in \mathbb{F}\right] = \operatorname{Span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

This example shows that the left-hand side of a linear equation of the type we started this course with is really a linear form, and the solution of a homogeneous equation is nothing but its null space. ▲▲▲

Example: Let $V = M_2(\mathbb{F})$ and let $\ell = \text{tr}$, i.e.,

$$\ell\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d.$$

It is easy to see that

$$\{\ell\}_0 = \left\{\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \;:\; a, b, c \in \mathbb{F}\right\},$$

or

$$\{\ell\}_0 = \text{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}.$$

▲▲▲

The following three propositions are the analogs of Propositions 4.11–4.13:

Proposition 4.15 *The null space of a set of linear forms is a vector subspace: let V be a vector space and let $L \subseteq V^\vee$, then*

$$L_0 \le V.$$

Proof: The set L_0 is non-empty because it contains 0_V. Let $\mathbf{u}, \mathbf{v} \in L_0$, i.e.,

$$\ell(\mathbf{u}) = \ell(\mathbf{v}) = 0_{\mathbb{F}} \qquad \text{for all } \ell \in L.$$

Then,

$$\ell(\mathbf{u} + \mathbf{v}) = \ell(\mathbf{u}) + \ell(\mathbf{v}) = 0_{\mathbb{F}} \qquad \text{for all } \ell \in L,$$

which implies that $\mathbf{u} + \mathbf{v} \in L_0$. For $\mathbf{u} \in L_0$ and $a \in \mathbb{F}$,

$$\ell(a\,\mathbf{u}) = a\,\ell(\mathbf{u}) = 0_{\mathbb{F}} \qquad \text{for all } \ell \in L,$$

which implies that $a\,\mathbf{u} \in L_0$. By definition, L_0 is a linear subspace of V. ∎

Proposition 4.16 *Let $(V, +, \mathbb{F}, \cdot)$ be a vector space and let $L, M \subseteq V^\vee$. Then,*

(a) If $L \subseteq M$ then $M_0 \le L_0$.

(b) $L_0 = (\text{Span}\, L)_0$

Proof: Before we prove it formally, two observations: (i) the larger a set of linear forms is, the more constraints are imposed on its null space, hence its null space should be smaller. (ii) Think of L_0 as a set of homogeneous linear equations on $\mathbb{F}^n_{\text{col}}$ (just as an example—we haven't even required V to be finitely-generated). The span of L is the set of all linear equations that are linear combinations of the equations in L; we know that the space of solutions doesn't change, which explains the second item.

And now to the formal proof. For the first item, let $\mathbf{v} \in M_0$, i.e.,

$$\ell(\mathbf{v}) = 0_{\mathbb{F}} \qquad \text{for all } \ell \in M.$$

Since $L \subseteq M$, it follows that

$$\ell(\mathbf{v}) = 0_{\mathbb{F}} \qquad \text{for all } \ell \in L,$$

i.e., $\mathbf{v} \in L_0$, proving that $M_0 \subseteq L_0$.

For the second item, let $\mathbf{v} \in L_0$, i.e.,

$$\ell(\mathbf{v}) = 0_{\mathbb{F}} \qquad \text{for all } \ell \in L.$$

Every $\ell \in \operatorname{Span} L$ is of the form

$$\ell = a_1\ell^1 + \cdots + a_n\ell^n$$

for some $\ell^1, \dots, \ell^n \in L$, hence

$$\ell(\mathbf{v}) = \left(a_1\ell^1 + \cdots + a_n\ell^n\right)(\mathbf{v}) = a_1\ell^1(\mathbf{v}) + \cdots + a_n\ell^n(\mathbf{v}) = 0_{\mathbb{F}},$$

proving that $\mathbf{v} \in (\operatorname{Span} L)_0$, i.e.,

$$L_0 \subseteq (\operatorname{Span} L)_0.$$

Conversely, since $L \subseteq \operatorname{Span} L$, it follows from the first item that $(\operatorname{Span} L)_0 \subseteq L_0$, proving that $(\operatorname{Span} L)_0 = L_0$. ■

Proposition 4.17 *Let $(V, +, \mathbb{F}, \cdot)$ be a finitely-generated vector space and let $L \leq V^\vee$. Then,*

$$\dim_{\mathbb{F}} L + \dim_{\mathbb{F}} L_0 = \dim_{\mathbb{F}} V.$$

Proof: This is left as an exercise; start with a basis for L_0. ■

We now combine the notions of null sets and annihilators to prove the following:

Proposition 4.18 *Let V be a finitely-generated vector space. Let $W \le V$ and let $L \le V^\vee$. Then,*

$$(W^0)_0 = W \qquad \textit{and} \qquad (L_0)^0 = L. \tag{4.1}$$

Proof: By Proposition 4.17 and Proposition 4.13,

$$\dim_{\mathbb{F}} W^0 + \dim(W^0)_0 = \dim_{\mathbb{F}} V$$

and

$$\dim_{\mathbb{F}} W + \dim_{\mathbb{F}} W^0 = \dim_{\mathbb{F}} V,$$

from which we conclude that W and $\dim(W^0)_0$ have the same dimension. It suffices then to show every vector in W is also in $(W^0)_0$ (actually, justify this assertion formally).

By definition,

$$(W^0)_0 = \{\mathbf{v} \in V \ : \ \ell(\mathbf{v}) = 0_{\mathbb{F}} \text{ for all } \ell \in W^0\},$$

whereas

$$W^0 = \{\ell \in V^\vee \ : \ \ell(\mathbf{w}) = 0_{\mathbb{F}} \text{ for all } \mathbf{w} \in W\}.$$

So let $\mathbf{w} \in W$. For every $\ell \in W^0$

$$\ell(\mathbf{w}) = 0_{\mathbb{F}},$$

from which follows that $\mathbf{w} \in (W^0)_0$, proving that $W \subseteq (W^0)_0$, which completes the proof. The second part is left as an exercise. ■

Corollary 4.19 *Let V be a finitely-generated vector space and let $U, W \le V$. Then,*

$$U = W \qquad \textit{if and only if} \qquad U^0 = W^0.$$

Likewise, let $L, M \le V^\vee$. Then,

$$L = M \qquad \textit{if and only if} \qquad L_0 = M_0.$$

Proof: We prove the first item. One direction is obvious, $U = W$ implies that $U^0 = W^0$. The other direction follows from the fact that $U^0 = W^0$ implies that $(U^0)_0 = (W^0)_0$, along with (4.1). The second item is left as an exercise. ■

Exercises

(intermediate) 4.15 Let $(V, +, \mathbb{F}, \cdot)$ be a vector space and let $W \leq V$. Define

$$U = \{\ell \in V^\vee \ : \ W \leq \{\ell\}_0\}.$$

Show that $U \leq V^\vee$.

(easy) 4.16 Let

$$\mathbf{w} = (1, 1) \in \mathbb{R}^2.$$

Calculate $\{\mathbf{w}\}^0$.

(intermediate) 4.17 Let $(V, +, \mathbb{F}, \cdot)$ be a finitely-generated vector space, let $W_1, W_2 \leq V$ and let $L_1, L_2 \leq V^\vee$. Show that

(a) $(W_1 \cap W_2)^0 = (W_1)^0 + (W_2)^0$.

(b) $(W_1 + W_2)^0 = (W_1)^0 \cap (W_2)^0$.

(c) $(L_1 \cap L_2)_0 = (L_1)_0 + (L_2)_0$.

(d) $(L_1 + L_2)_0 = (L_1)_0 \cap (L_2)_0$.

(intermediate) 4.18 Find a basis for the annihilator of

$$W = \operatorname{Span}\left((1, 2, -3, 4), (0, 1, 4, -1)\right) \leq \mathbb{R}^4.$$

(intermediate) 4.19 Let $V = (\mathbb{R}^4, +, \mathbb{R}, \cdot)$, and let

$$\ell_1(\mathbf{x}) = x^1 + 2x^2 + 2x^3 + x^4 \qquad \ell_2(\mathbf{x}) = 2x^1 + x^4$$

$$\ell_3(\mathbf{x}) = -2x^1 - 3x^3 + 3x^4.$$

Find a subspace $W \leq \mathbb{R}^4$ such that

$$W^0 = \operatorname{Span}(\{\ell_1, \ell_2, \ell_3\}).$$

(intermediate) 4.20 Let V be a finitely-generated vector space and let $L \leq V^\vee$. Show that

$$(L_0)^0 = L.$$

Conclude that for $L, M \leq V^\vee$,

$$L = M \qquad \text{if and only if} \qquad L_0 = M_0.$$

(harder) 4.21 Prove Proposition 4.17.

4.5.3 Linear systems and linear forms

Let $A \in M_{m\times n}(\mathbb{F})$. We consider the space of solutions

$$S_A = \{\mathbf{v} \in \mathbb{F}^n_{\text{col}} \ : \ A\mathbf{v} = 0_{\mathbb{F}^m_{\text{col}}}\}$$

of the homogeneous linear system. Each of the m rows of A can be viewed as a linear form acting on an element of $\mathbb{F}^n_{\text{col}}$; Thus the set of solutions S_A equals,

$$S_A = \{\mathbf{v} \in \mathbb{F}^n_{\text{col}} \ : \ \text{Row}^i(A)\mathbf{v} = 0, \ i = 1, \dots, m\} = \{\text{Row}^i(A) \ : \ i = 1, \dots, m\}_0.$$

By Proposition 4.16,

$$S_A = \big(\text{Span}\{\text{Row}^i(A) \ : \ i = 1, \dots, m\}\big)_0 = (\mathscr{R}(A))_0,$$

i.e., the set of solutions is the null space of the row space of A. Proposition 4.17 asserts that

$$\dim_{\mathbb{F}} \mathscr{R}(A) + \dim_{\mathbb{F}} S_A = \dim_{\mathbb{F}} \mathbb{F}^n_{\text{col}} = n.$$

Recall that the dimension of the row space equals the dimension of the column space, and that this dimension is called the rank of the matrix. Thus,

$$\dim_{\mathbb{F}} S_A = n - \text{rank} A.$$

In other words, for a homogeneous linear system of m equations in n unknowns, the space of solutions is a linear subspace of $\mathbb{F}^n_{\text{col}}$, whose dimension is n minus the rank of A, which we recall is the number of non-zero rows in its row-reduced form (make sure that this makes sense to you).

Example: Consider once again the matrix

$$\begin{bmatrix} 0 & 0 & 1 & 4 \\ 2 & 4 & 2 & 6 \\ 3 & 6 & 2 & 5 \end{bmatrix},$$

whose row-reduced form is

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In this case, $n = 4$ and $\operatorname{rank} A = 2$. As for the space of solutions, its dimension is 2,

$$S_A = \left\{ \begin{bmatrix} -2s+t \\ s \\ -4t \\ t \end{bmatrix} \ : \ s,t \in \mathbb{R} \right\} = \operatorname{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}.$$

▲▲▲

Example: Let's have a different look on the relation between equations and solutions. Let $V = \mathbb{F}^3_{\text{col}}$; then $V^\vee = \mathbb{F}^3_{\text{row}}$ under the action through row-column multiplication. We use the standard bases for V and $V^\vee$. Consider the linear form

$$\ell(\mathbf{x}) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} = x^1 + x^2 + x^3.$$

The space of solutions, which is the null space of $\{\ell\}$ is

$$\{\ell\}_0 = \left\{ \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} \ : \ s,t \in \mathbb{F} \right\} = \operatorname{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \leq \mathbb{F}^3_{\text{col}}.$$

The equation represented by the linear form whose coordinates (relative to the standard dual basis) are $[1,1,1]$, induces a space of solutions, which is a two-dimensional subspace of $\mathbb{F}^3_{\text{col}}$. As we know, the space of solution does not change if we multiply ℓ by any non-zero scalar: the space of solution is in fact the null space of the one-dimensional subspace of linear forms, whose coordinate representation is

$$\mathbb{F}[1,1,1] = \{[a,a,a] \ : \ a \in \mathbb{F}\}.$$

Denote the space of solutions by W. We may ask the opposite question: does the space of solutions define the equation whose solution they are. This is really asking: what are all the linear forms ℓ satisfying $\ell(\mathbf{w}) = 0_{\mathbb{F}}$ for all $\mathbf{w} \in W$. Write such a linear form as

$$\ell = a_1 \,\mathbf{e}^1 + a_2 \,\mathbf{e}^2 + a_3 \,\mathbf{e}^3,$$

we require that $\ell \in W^0$, which is the case if and only if

$$\ell\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right) = -a_1 + a_2 = 0_{\mathbb{F}} \qquad \text{and} \qquad \ell\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right) = -a_1 + a_3 = 0_{\mathbb{F}},$$

from which we obtain that $a_1 = a_2 = a_3$, i.e., ℓ must be of the form

$$\ell = a\,(\mathbf{e}^1 + \mathbf{e}^2 + \mathbf{e}^3) = a\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{e}^1 \\ \mathbf{e}^2 \\ \mathbf{e}^3 \end{bmatrix},$$

which is what we expected. This examples show once again the relations between equations and solutions as linear subspaces of vectors and linear forms. ▲ ▲ ▲