Chapter 5

L^p spaces

5.1 Basic theory

Throughout this chapter, we assume a fixed measure space $(\mathbb{X}, \Sigma, \mu)$.

Definition 5.1 *Let $f : \mathbb{X} \to \mathbb{R}$ be measurable. For $p \in (0, \infty)$ we define*

$$\|f\|_p = \left(\int_{\mathbb{X}} |f|^p \, d\mu \right)^{1/p}.$$

(This expression can be infinite.)

Definition 5.2 *Let $p \in (0, \infty)$.*

$$L^p(\mu) = \{ f : \mathbb{X} \to \mathbb{R} \;:\; f \text{ is measurable and } \|f\|_p < \infty. \}$$

$L^p(\mu)$-spaces are generalizations of $L^1(\mu)$-spaces; the elements of $L^p(\mu)$ are equivalence classes of functions, which may differ on sets of zero measure.

Lemma 5.3 *$L^p(\mu)$ is a vector space.*

Proof: Closure under scalar multiplication is obvious. Closure under addition follows from the inequality

$$|f+g|^p \le (2 \max(|f|, |g|))^p \le 2^p (|f|^p + |g|^p),$$

hence

$$\|f+g\|_p^p \le 2^p \left(\|f\|_p^p + \|g\|_p^p\right)$$

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We next show that as the notation indicates, $\|\cdot\|_p$ is a norm on $L^p(\mu)$ (for $p \ge 1$).

Definition 5.4 Let $p > 1$. We denote its conjugate exponent (חזקה צמודה) by

$$p^* = p/(p-1).$$

Note that

$$\frac{1}{p} + \frac{1}{p^*} = 1.$$

Lemma 5.5 (Young's inequality) Let $p > 1$ and set $q = p^*$. Then, for every $a, b \in \mathbb{R}$:

$$|ab| \le \frac{|a|^p}{p} + \frac{|b|^q}{q}$$

Proof: Since $(-\log)$ is a convex function and since $1/p$ and $1/q$ sum up to one, then, for every $\alpha, \beta > 0$:

$$-\log\left(\frac{\alpha}{p} + \frac{\beta}{q}\right) \le -\frac{1}{p}\log\alpha - \frac{1}{q}\log\beta = -\log(\alpha^{1/p}\beta^{1/q}).$$

It follows that:

$$\frac{\alpha}{p} + \frac{\beta}{q} \ge \alpha^{1/p}\beta^{1/q}.$$

Setting $\alpha = |a|^p$ and $\beta = |b|^q$ we recover the desired result. ■

Proposition 5.6 (Hölder inequality) Let $p > 1$ and set $q = p^*$. For every measurable $f, g : \mathbb{X} \to \mathbb{R}$,

$$\|fg\|_1 \le \|f\|_p \|g\|_q.$$

In particular, if $f \in L^p(\mu)$ and $q \in L^q(\mu)$, then $fg \in L^1(\mu)$.

Proof: If either $\|f\|_p = 0$ or $\|g\|_q = 0$, then the result is trivial. Otherwise, using Young's inequality,

$$\left\| \frac{fg}{\|f\|_p \|g\|_q} \right\|_1 = \int_{\mathbb{X}} \frac{f}{\|f\|_p} \frac{g}{\|g\|_q} \, d\mu \leq \frac{1}{p} \int_{\mathbb{X}} \frac{|f|^p}{\|f\|_p^p} \, d\mu + \frac{1}{q} \int_{\mathbb{X}} \frac{|g|^q}{\|g\|_q^q} \, d\mu = \frac{1}{p} + \frac{1}{q} = 1.$$

Multiplying both sides by $\|f\|_p \|g\|_q$ we obtain the desired result. ■

Proposition 5.7 (Minkowski inequality) Let $p \geq 1$. For every $f, g \in L^p(\mu)$,

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof: The inequality is trivial for $p = 1$. For $p > 1$ set $q = p^*$; it follows from the triangle inequality that

$$|f + g|^p = |f + g| \, |f + g|^{p-1} \leq |f| \, |f + g|^{p-1} + |g| \, |f + g|^{p-1}.$$

Integrating, and using Hölder's inequality:

$$\|f + g\|_p^p \leq \|f\|_p \|f + g\|_p^{p-1} + \|g\|_p \|f + g\|_p^{p-1},$$

where we used the defining property of q. This completes the proof. ■

Corollary 5.8 For every $p \geq 1$, $\|\cdot\|_p$ is a norm on $L^p(\mu)$.

Proof: Positivity and homogeneity are immediate; the triangle inequality is nothing but Minkowski's inequality. ■

✎ *Exercise 5.1* Show that $\|\cdot\|_p$ is not a norm for $0 < p < 1$.

*Proposition 5.9 $L^p(\mu)$ is a complete normed space, i.e., a **Banach space** (מרחב בנך).*

✎ *Exercise 5.2* Prove the completeness of $L^p(\mu)$ in two steps:

(a) Prove that it suffices to prove that for every sequence $f_n \in L^p(\mu)$,

$$\text{if} \quad \sum_{n=1}^{\infty} \|f_n\|_p < \infty \qquad \text{then} \qquad \sum_{n=1}^{\infty} f_n \quad \text{converges in } L^p(\mu).$$

(b) Prove the statement by showing that $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$ implies that $F \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} f_n$ exists a.e.; then show that $F \in L^p(\mu)$ and that convergence is in $L^p(\mu)$.

Proposition 5.10 For every $p \geq 1$, the space of simple functions $f = \sum_j a_j \chi_{E_j}$ for which $\mu(E_j) < \infty$ for all j is dense in $L^p(\mu)$.

Proof: Let $f \in L^p(\mu)$. Then, there exist simple functions $\phi_n \in \mathrm{SF}^+(\mathbb{X})$ converging monotonically to f^+, and simple functions $\psi_n \in \mathrm{SF}^+(\mathbb{X})$ converging monotonically to f^-. Set $f_n = \phi_n - \psi_n$. Then, $f_n \to f$ pointwise, and

$$|f_n| \leq \phi_n + \psi_n \leq f^+ + f^- = |f|.$$

It follows that $f_n \in L^p(\mu)$, hence the E_j in the representation of f_n satisfy $\mu(E_j) < \infty$.

Furthermore,

$$|f_n - f|^p \leq (|f_n| + |f|)^p \leq 2^p |f|^p,$$

i.e., $|f_n - f|^p \in L^1(\mu)$. It follows from dominated convergence that

$$\lim_{n\to\infty} \int_{\mathbb{X}} |f_n - f|^p \, d\mu = 0.$$

■

The spaces L^p are generalizations of the space $L^1(\mu)$. In general, there is no inclusion relations between those space:

Proposition 5.11 In general, for every $p \neq q$ there is no inclusion relation between $L^p(\mu)$ and $L^q(\mu)$.

Proof: Consider the space $([1,\infty),\mathscr{B}([1,\infty)),m)$ and set $f(x)=1/x$. Then,

$$f\in L^2(m) \qquad \text{but} \qquad f\notin L^1(m).$$

Consider the space $([-1,1),\mathscr{B}([-1,1)),m)$ and set $g(x)=1/\sqrt{x}$. Then,

$$g\in L^1(m) \qquad \text{but} \qquad g\notin L^2(m).$$

■

However, we have the following results:

Proposition 5.12 *Let* $1<p<q<r$ *. Then,*

$$L^q(\mu)\subset L^p(\mu)+L^r(\mu).$$

That is, every $f\in L^q(\mu)$ *can be represented as* $g+h$*, where* $g\in L^p(\mu)$ *and* $h\in L^r(\mu)$*.*

Proof: Let $f\in L^q(\mu)$ be given and let

$$E=\{x\,:\,|f(x)|>1\}.$$

Define $g=f\chi_E$ and $h=f\chi_{E^c}$. Then, $f=g+h$, and

$$|g|^p=|f|^p\chi_E\le|f|^q\chi_E \qquad \text{and} \qquad |h|^r=|f|^r\chi_{E^c}\le|f|^q\chi_{E^c},$$

which proves that

$$\|g\|_p\le\|f\|_q \qquad \text{and} \qquad \|h\|_r\le\|f\|_q.$$

■

Proposition 5.13 *Let* $1<p<q<r$ *. Then,*

$$L^p(\mu)\cap L^r(\mu)\subset L^q(\mu).$$

Proof: Let $f \in L^p(\mu) \cap L^r(\mu)$. Since $q/p > 1$ and $q/r < 1$, there exists a $\lambda \in (0,1)$ satisfying

$$\frac{\lambda q}{p} + \frac{(1-\lambda)q}{r} = 1.$$

Using Hölder's inequality

$$\begin{aligned}\|f\|_q^q = \int_{\mathbb{X}} |f|^q \, d\mu = \int_{\mathbb{X}} |f|^{\lambda q}|f|^{(1-\lambda)q} \, d\mu = \left\| |f|^{\lambda q}|f|^{(1-\lambda)q} \right\|_1 \\ \leq \left\| |f|^{\lambda q} \right\|_{p/\lambda q} \left\| |f|^{(1-\lambda)q} \right\|_{r/(1-\lambda)q} = \|f\|_p^{\lambda q} \|f\|_r^{(1-\lambda)q},\end{aligned}$$

i.e., $\|f\|_q \leq \|f\|_p^{\lambda}\|f\|_r^{1-\lambda} < \infty$. ■

If the measure space is finite, then the $L^p(\mu)$ space satisfy the following inclusion relation:

Proposition 5.14 *If $\mu(\mathbb{X}) < \infty$ then $1 \leq p < q$ implies that $L^q(\mu) \subset L^p(\mu)$.*

Proof: Let $f \in L^q(\mu)$ and let

$$E = \{x \, : \, |f(x)| \leq 1\}.$$

Then,

$$\int_{\mathbb{X}} |f|^p \, d\mu = \int_E |f|^p \, d\mu + \int_{E^c} |f|^p \, d\mu \leq \mu(E) + \int_{E^c} |f|^q \, d\mu < \infty.$$

■

TA material 5.1 Define and treat the case of $p = \infty$.

✎ *Exercise 5.3* Let $(\mathbb{X}, \Sigma, \mu)$ be a finite measure space, let $f \in L^p(\mu)$ and let $q = p^*$. Show that

$$\|f\|_1 \leq (\mu(\mathbb{X}))^{1/q}\|f\|_p.$$

✎ *Exercise 5.4* Consider the measure space $([0,1], \mathscr{B}([0,1]), m)$ and let $f \in L^p(m)$ for some $p > 1$. Let $q = p^*$. Prove that

$$\lim_{t \searrow 0} \frac{1}{t^{1/q}} \int_{[0,t]} |f| \, dm = 0.$$

✎ *Exercise 5.5* Let $(\mathbb{X}, \Sigma, \mu)$ be a finite measure space. Let $f_n, f \in L^2(\mu)$, such that

$$\|f_n\|_2 \le M \qquad \text{and} \qquad \lim_{n\to\infty} f_n = f \ \text{ a.e.}$$

(a) Prove that $f \in L^2$. (b) Use Egorov's theorem to show that $f_n \to f$ in $L^1(\mu)$.

✎ *Exercise 5.6* Let μ be the counting measure on $\mathbb{N}$ and let $1 < p < q, \infty$. Show that

$$\|f\|_p \le \|f\|_q.$$

5.2 Duality

*Definition 5.15 Let $p \ge 1$. A **bounded linear functional** (פונקציונל ליניארי חסום) on $L^p(\mu)$ is a linear mapping $\phi : L^p(\mu) \to \mathbb{R}$, satisfying*

$$\|\phi\| \stackrel{def}{=} \sup\{|\phi(f)| \,:\, \|f\|_p = 1\} < \infty.$$

*This space is called the space **dual** (דואלי) to $L^p(\mu)$ and is denoted $(L^p(\mu))^*$.*

The mapping $\phi \mapsto \|\phi\|$ is the **operator norm** (נורמה אופרטורית); you must have seen in the past that it is indeed a norm on $(L^p(\mu))^*$.

Also,

Proposition 5.16 Bounded linear functionals on $L^p(\mu)$ are continuous.

Proof: Let $f_n \to f$ in $L^p(\mu)$ and let $\phi \in (L^p(\mu))^*$. Then,

$$|\phi(f_n) - \phi(f)| = |\phi(f_n - f)| = \left|\phi\left(\frac{f_n - f}{\|f_n - f\|_p}\right)\right| \|f_n - f\|_p \le \|\phi\| \|f_n - f\|_p \to 0.$$

■

Proposition 5.17 Let $p > 1$ and $q = p^*$ be its conjugate. Then, for every $g \in L^q(\mu)$, the map

$$\phi_g : f \mapsto \int_{\mathbb{X}} f g \, d\mu$$

is a bounded linear functional. Moreover,

$$\|\phi_g\| = \|g\|_q.$$

Proof: Clearly, ϕ_q is a linear functional. Let $\|f\|_p = 1$. By Hölder's inequality,

$$|\phi_g(f)| \le \int_{\mathbb{X}} |fg|\, d\mu \le \|f\|_p \|g\|_q = \|g\|_q,$$

from which follows that $\|\phi\|_g \le \|g\|_q$. Equality is obtained by taking,

$$f = \operatorname{sgn}(g) \frac{|g|^{q-1}}{\|g\|_q^{q-1}},$$

in which case

$$\|f\|_p = \int_{\mathbb{X}} \frac{|g|^{p(q-1)}}{\|g\|_q^{p(q-1)}}\, d\mu = \int_{\mathbb{X}} \frac{|g|^q}{\|g\|_q^q}\, d\mu = 1,$$

and

$$|\phi_g(f)| = \int_{\mathbb{X}} \operatorname{sgn}(g) \frac{|g|^{q-1}}{\|g\|_q^{q-1}} g\, d\mu = \int_{\mathbb{X}} \frac{|g|^q}{\|g\|_q^{q-1}}\, d\mu = \|g\|_q.$$

■

In other words, the mapping

$$L^q(\mu) \to (L^p(\mu))^*,$$

which maps $g \in L^q(\mu)$ into $\phi_g \in (L^p(\mu))^*$ is an **isometric embedding** (שיכון איזומטרי). The following seminal theorem shows that this map is in fact an isometry.

Theorem 5.18 (Riesz representation for L^p-spaces) Let $p > 1$ and let $q = p^*$. Then

$$L^q(\mu) \simeq (L^p(\mu))^*$$

in the category of Banach spaces. That is, to every $\phi \in (L^p(\mu))^*$ corresponds a unique $g \in L^q(\mu)$ such that $\phi = \phi_g$ and $\|\phi\| = \|g\|_q$.

Proof: We will only prove the theorem for the case of a finite measure space. It remains to prove that to every ϕ corresponds a unique g such that $\phi = \phi_g$.

Step 1: uniqueness: If $\phi_g = \phi_h$, then for every $f \in L^p(\mu)$,

$$\int_{\mathbb{X}} fg\, d\mu = \int_{\mathbb{X}} fh\, d\mu,$$

hence for every $f \in L^p(\mu)$,

$$0 = \int_{\mathbb{X}} f(g-h)\, d\mu.$$

Suppose that $g \neq h$. Then, one of the sets

$$\{x \,:\, g(x) > f(x)\} \qquad \text{or} \qquad \{x \,:\, g(x) < f(x)\}.$$

has positive measure. Without loss of generality, we may assume that this is the first. Then, for some n,

$$E_n = \left\{ x \,:\, g(x) - h(x) > \frac{1}{n} \right\}$$

has positive measure. Setting $f = \chi_{E_n}$ we obtain a contradiction.

Step 2: construct a signed-measure ν: Let $\phi \in (L^p(\mu))^*$ be given. Since μ is finite, $\chi_E \in L^p(\mu)$ for every measurable set $E \in \Sigma$; define

$$\nu(E) = \phi(\chi_E).$$

We will show that ν is a signed measure:

(a) Since $\chi_\varnothing = 0$ as an element of $L^p(\mu)$, and ϕ is linear,

$$\nu(\varnothing) = \phi(\chi_E) = \phi(0) = 0.$$

(b) Let

$$E = \coprod_{n=1}^{\infty} E_n,$$

then

$$\lim_{n\to\infty} \sum_{k=1}^{n} \chi_{E_k} = \chi_E \qquad \text{pointwise,}$$

and

$$\left\| \chi_E - \sum_{k=1}^{n} \chi_{E_k} \right\|_p = \left\| \sum_{k=n+1}^{\infty} \chi_{E_k} \right\|_p = \left(\int_{\coprod_{k=n+1}^{\infty} E_k} d\mu \right)^{1/p} = \left(\mu\left(\coprod_{k=n+1}^{\infty} E_k \right) \right)^{1/p}.$$

Letting $n \to \infty$, using the upper-semicontinuity of μ and the fact that $\bigcap_{n=1}^{\infty} \coprod_{k=n+1}^{\infty} E_k = \varnothing$,

$$\lim_{n\to\infty} \left\| \chi_E - \sum_{k=1}^{n} \chi_{E_k} \right\|_p = \lim_{n\to\infty} \left(\mu\left(\coprod_{k=n+1}^{\infty} E_k \right) \right)^{1/p} = \left(\mu\left(\bigcap_{n=1}^{\infty} \coprod_{k=n+1}^{\infty} E_k \right) \right)^{1/p} = 0,$$

i.e.,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\chi_{E_k} = \chi_E \qquad \text{in } L^p(\mu).$$

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Since ϕ is linear and continuous,

$$\nu(E) = \phi(\chi_E) = \sum_{n=1}^{\infty}\phi(\chi_{E_n}) = \sum_{n=1}^{\infty}\nu(E_n).$$

proving that ν is countably-additive, i.e., it is a signed measure.

Step 3: Show that $\nu \ll \mu$ and apply Radon-Nikodym to obtain a function g: Suppose that $\mu(E) = 0$. Then, $\chi_E = 0$ (as an element of $L^p(\mu)$), hence

$$\nu(E) = \phi(\chi_E) = 0.$$

It follows from the Radon-Nikodym theorem that there exists an integrable function $g \in L^1(\mu)$, such that

$$\nu(E) = \int_E g\,d\mu,$$

which amounts to

$$\phi(\chi_E) = \int_{\mathbb{X}} g\chi_E\,d\mu.$$

By linearity,

$$\phi(\psi) = \int_{\mathbb{X}} g\,\psi\,d\mu \qquad \text{for all simple functions } \psi : \mathbb{X} \to \mathbb{R}.$$

By definition of the norm $\|\phi\|$,

$$|\phi(\psi)| = \left|\int_{\mathbb{X}} g\,\psi\,d\mu\right| \le \|\phi\|\,\|\psi\|_p \qquad \text{for all simple functions } \psi : \mathbb{X} \to \mathbb{R}.$$

Step 4: Prove that $g \in L^q(\mu)$: (Recall that $L^q(\mu) \subset L^1(\mu)$, but not the other way around.) Let g_n be a sequence of simple functions converging to g pointwise, such that $|g_n| \le |g|$. By Fatou's lemma for $|g_n| \to |g|$,

$$\|g\|_q \le \liminf_{n\to\infty}\|g_n\|_q.$$

Set

$$f_n = \operatorname{sgn}(g)\frac{|g_n|^{q-1}}{\|g_n\|_q^{q-1}},$$

which is a sequence of simple functions satisfying $\|f_n\|_p = 1$. Then,

$$
\begin{aligned}
\|g_n\|_q &= \frac{1}{\|g_n\|^{q-1}} \int_{\mathbb{X}} |g_n|^q \, d\mu \\
&= \int_{\mathbb{X}} |f_n g_n| \, d\mu \\
&\le \int_{\mathbb{X}} |f_n g| \, d\mu \\
&= \int_{\mathbb{X}} f_n g \, d\mu \\
&= \phi(f_n) \le \|\phi\|.
\end{aligned}
$$

In the passage to the second line we used the definition of f_n; in the passage to the third line we used the fact that $|g_n| \le |g|$; in the passage to the fourth line we used the fact that $f_n g > 0$; in the passage to the fifth line we used the characterization of ϕ for simple functions; in the passage to the sixth line we used the definition of the operator norm and the fact that $\|f_n\|_p = 1$.

It follows that

$$
\|g\|_q \le \liminf_{n\to\infty} \|g_n\|_q \le \|\phi\|.
$$

Step 5: Prove that $\phi \in \phi_g$: Finally, let $f \in L^p(\mu)$. Since the space of simple functions is dense in $L^p(\mu)$ and ϕ is continuous, for $\psi_n \to f$ in $L^p(\mu)$,

$$
\begin{aligned}
\left| \int_{\mathbb{X}} g f \, d\mu - \phi(f) \right| &= \lim_{n\to\infty} \left| \int_{\mathbb{X}} g f \, d\mu - \phi(\psi_n) \right| \\
&= \lim_{n\to\infty} \left| \int_{\mathbb{X}} g f \, d\mu - \int_{\mathbb{X}} g \psi_n \, d\mu \right| \\
&= \lim_{n\to\infty} \left| \int_{\mathbb{X}} g (f - \psi_n) \, d\mu \right| \\
&\le \lim_{n\to\infty} \|g\|_q \|f - \psi_n\|_p = 0.
\end{aligned}
$$

This completes the proof for the case where μ is a finite measure. ■ —36h(2017)—